Tight single-change covering designs

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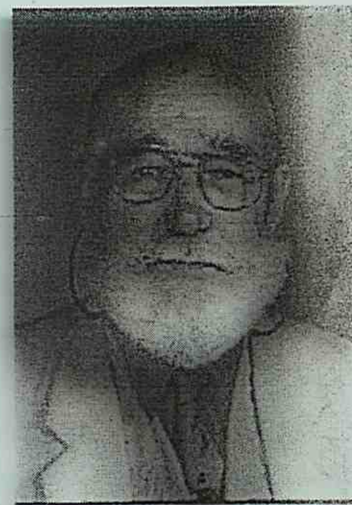
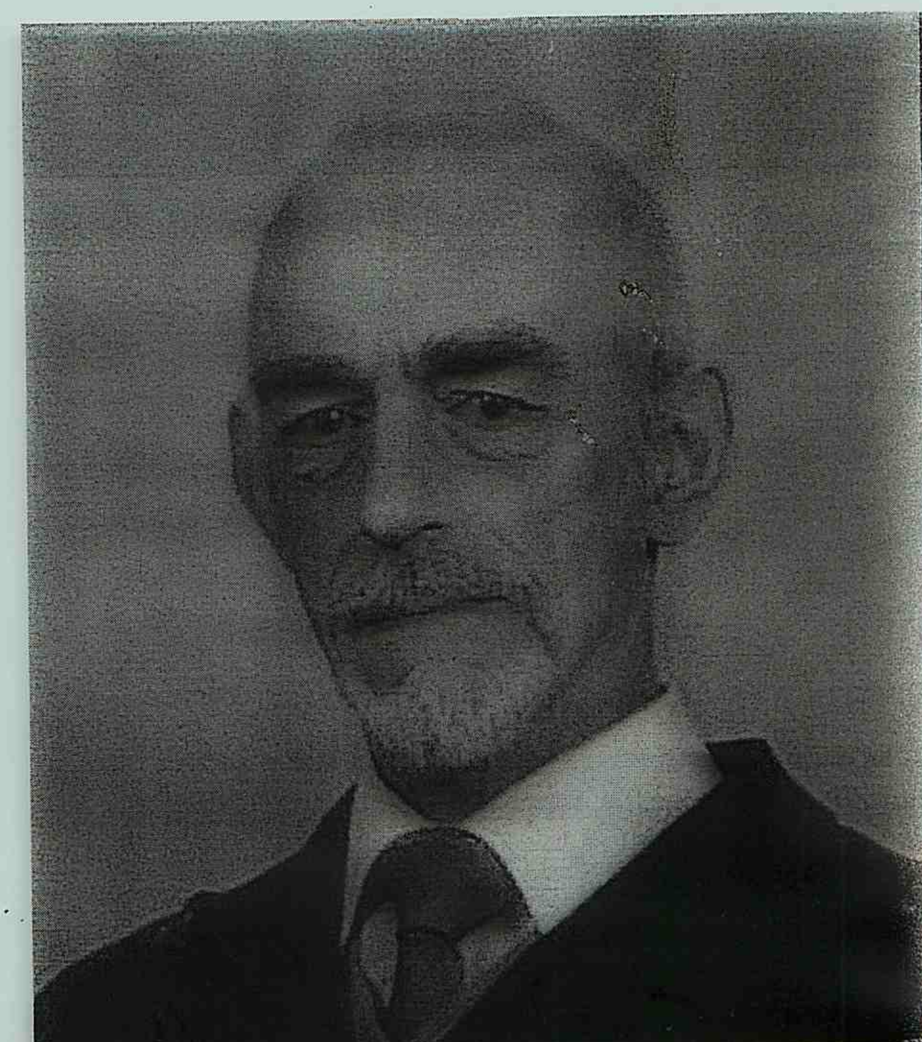
- History and progenitors of the problem
- A design with block size 3
- Possible values for the number of blocks
- An infinite family of designs of block size 4.
- Algorithmic approaches
- A design with block size 5
- The open problem; untested approaches.

John Ashcroft Nelder, FRS
(1924-)



Head of Statistics and Biomathematics at Rothamsted Experimental Station, 1968-1984
Now at Imperial College, London



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In 1968, computers had two kinds of memory.

- Small central-access stores
- Large backing stores

Nelder wanted to calculate $v \times v$ sums-of-squares-and-products matrices and correlation matrices when only $k < v$ of the variates could fit into the immediate-access store ~~at~~ one time.

- v sums of squares required (k accessible)
- $\binom{v}{2}$ sums of products required ($\binom{k}{2}$ accessible)
- $\binom{v}{2}$ correlation coefficients ($\binom{k}{2}$ accessible)

$\therefore$ start with k variates in immediate access store and change one at a time until every pair has occurred twice.

- Single-change covering designs.
  - Each pair of entries occurs in at least one block "covering"
  - When each block is formed from the previous one, only a single change is made.
  - When each new block is formed, the new entry occurs only with entries with which it has not been previously paired, so the arrangement is "tight".

1	1	1	1	1	3	3	3	2	2
2	2	5	6	7	7	7	7	7	6
3	4	4	4	4	4	6	5	5	5

This design has $v=7$, $k=3$, $b=10$.

There are $T=12$ transfers.

Row-regularity. The number of transfers is the same for each row.

Element-regularity. Each element is transferred the same number of times.

1	1	1	1	3	3	2	←	
2	2	5	6	6	6	6	←	row regular.
3	4	4	4	4	5	5	←	

1	1	1	1	1	1	3	3	3	3	2	2	1	7	7	7	6	6	6	6	6	
2	2	2	2	2	9	9	9	9	9	9	10	10	10	10	10	10	9	11	8	8	← r-r.
3	3	3	7	7	7	7	10	11	11	11	11	11	11	4	4	4	4	4	4	12	+ e-r.
4	5	6	6	8	8	8	8	8	12	12	12	12	12	12	5	5	5	5	5	5	

(Constable)

Standard form.

- Elements of the first block are ordered $1, 2, \dots k$
- Other elements are initially introduced in the order $k+1, k+2, \dots, v$.
- Elements of the first block are changed initially in the order $k, k-1, \dots, 2, 1$.

Wallis' Problem.

An electrical engineer at S. Illinois University had a problem involving v electrical components that had to be paired in testing, with just k of them wired up simultaneously.

Possible pairs of values of v and k.

- Number of pairs of integers $1, 2, \ldots v$ is $\binom{v}{2}$.
- Number of pairs in first block of a tsccd(v,k) is $\binom{k}{2}$.
- Number of NEW pairs in each subsequent block is $k-1$.

$$\binom{v}{2} = \binom{k}{2} + (b-1)(k-1). \qquad (1)$$

$$b = \frac{\binom{v}{2} - \binom{k}{2}}{k-1} + 1.$$

Thus $k-1$ divide $\binom{v}{2}-\binom{k}{2}$.

For $k=3$, $b = \dfrac{\binom{v}{2}-3}{2} + 1$.

$$\therefore \quad 2 \mid \binom{v}{2} - 3.$$

Thus $\binom{v}{2}$ is odd, so $v = 6, 7, 10, 11, 14, 15 \ldots$

For $k=4$, $b = \dfrac{\binom{v}{2}-6}{3} + 1$.

$$\therefore \quad 3 \mid \binom{v}{2} - 6.$$

Thus $3 \mid \binom{v}{2}$, So $v = 6, 7, 9, 10, 12, 13 \ldots$

For $k=5$, $v = 12, 13, 20, 21, 28, 29 \ldots$

For $k=6$, $v = 10, 11, 15, 16, 20, 21 \ldots$

By considering the number of transfers,

$$v \geq 3(k-1) \text{ and } v > 3k-2 \text{ for } k>3.$$

Thus a tsccd$(v,4)$ has $v \geq 12$ and a tsccd$(v,5)$ or tsccd$(v,6)$ has $v \geq 20$.

Finally $v(v-1) \geq (6v-7k)(k-1)$

which means that when $k=6$,

$$v(v-1) \geq 5(6v-42)$$

$$v^2 - v \geq 30v - 210$$

$$v^2 - 31v + 210 \geq 0$$

which isn't true for $v=20$.

Classification methods.

t_i the number of elements in a tsccd that are transferred i times.

f_i the number of blocks in a tsccd that contain any particular element that is transferred i times.

p_i the number of pairs of elements such that each pair occurs together in i successive blocks.

S_j number of transfers in a row j of a standardized tsccd(v,k).

Persistent pairs. Pairs which occur together in $v-k$ successive blocks.

Conversion of row-regular tsccd(12, 4) to tsccd(13, 4)

1	1	1	1	1	1	1	1	1	1	7	7	7	7	7	8	8	8	8	8	8	8	5	5	5
2	2	2	2	2	8	9	9	11	11	11	11	11	11	11	11	11	11	12	12	12	12	12	12	13
3	3	3	3	7	7	7	10	10	10	10	10	10	10	10	10	10	9	9	9	9	9	9	6	6
4	13	5	6	6	6	6	6	6	12	12	13	3	4	5	5	2	2	2	13	3	4	4	4	4

$\text{tsccd}(v', 3) \rightarrow \text{tsccd}(v'+4, 3)$.

A recursive construction can be used to obtain a tsccd$(12+3i, 4)$ for any $i = 1, 2, \ldots$

Algorithmic approaches.

- Hill-climbing
- Genetic algorithms
- Back-track searches.

A design with $k=5$.

$v=20, \; k=5, \; b=46.$

The minimum v possible for $k=6$ is $v=21$. We can have

$v = 21, 25, 26, 30, 31, 35, 36 \ldots$

Other approaches

- Integer programming with local and tabu searches.

WSAT (OIP)