"Integer Programming and Critical Sets"

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Integer Programming

- attempting to maximize or minimize an "objective function" subject to a set of "constraints"
- without the objective function, known as "constraint programming"
- all variables integral/binary ⇒ "0-1 IP"

Example: minimize $\sum_{i=0}^{2} \sum_{j=0}^{2} x_{ij}$

subject to: $i=0,1,2 \quad x_{i0} + x_{i1} + x_{i2} \geq 1$

$j=0,1,2 \quad x_{0j} + x_{1j} + x_{2j} \geq 1$

$k=0,1,2 \quad x_{0(0-k)} + x_{1(1-k)} + x_{2(2-k)} \geq 1$

This is also known as a "set covering" problem - what is the smallest set which intersects every set in a given collection of sets? It is known to be NP-hard.

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

constraints based on trades in $\mathbb{Z}_2^3$.

e.g. $$\sum_{i=0}^{1} \sum_{j=0}^{1} x_{ij} \geq 1$$

$$\sum_{i=0}^{3} \sum_{j=0}^{3} x_{ij} \geq 5$$

Answer is 25.

Similar constraints for $\mathbb{Z}_2^4$, except too many trades to find a provably optimal solution.

CPLEX takes several hours to find a solution to the IP of size 118, WSAT (free!) finds solutions of size 112 instantly.

Smallest CS found size 121, improving Khodkar's bound of 124.

General ~~lower~~ upper bound on size of smallest critical sets in $\mathbb{Z}_2^n$.

·	·	·	·	·	·	·	·
·	1	·	3	·	5	·	7
·	·	1	2	·	·	5	6
·	3	2	1	·	7	6	5
·	·	·	·	1	2	3	4
·	5	·	7	2	1	4	3
·	·	5	6	3	4	1	2
·	7	6	5	4	3	2	1

Improving Donovan et al's bound, we now have:

$$scs(\mathbb{Z}_2^n) \leq 4^n - 3^n + 4 - 2^n - 2^{n-2}$$

$\mathbb{Z}_3^2$

123	456	789
231	564	897
312	645	978
456	789	123
564	897	231
645	978	312
789	123	456
897	231	564
978	312	645

324 trades of size 6. Lots of symmetry. We can force entries into the top LH 3×3 subsquare by row and column swaps, so that it contains $\geq$ the number of entries in the other 8 3×3 subsquares.

$$24 \leq scs(\mathbb{Z}_3^2) \leq 29$$

Improving Donovan Cooper Nott and Seberry's result. (1995)