Subsquare-rich Latin squares and their large critical sets

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Latin square

- An n times n square with elements from a set of size n - say {0, …, n-1} - in each position, such that each row and each column contains each element exactly once.
- A **subsquare** S of a Latin square L is a subset of the rows and columns of L such that S is itself a Latin square.

Unique completion

- A **partial Latin square** is an n times n square array with each element occurring at most once in each row and each column.
- A PLS P is **uniquely completable** if exactly one Latin square is a superset of S.

Critical Sets

- A **critical set** is a uniquely completable set such that any subset is no longer uniquely completable.
- An **n-critical set** is a critical set such that each entry is in a n x n subsquare of the Latin square.
- A 2x2 subsquare is known as an **intercalate**.
- **lcs(n)** is the size of the largest critical set in a Latin square of order n.

Examples

	1		3
		1	2
	3	2	1

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Intercalate results

- The maximum number of intercalates in an n times n Latin squares is denoted **I(n)**.
- Heinrich and Wallis (1980) found:

$I(2n) \geq n^3$

$I(n) \leq \frac{n^2(n-1)}{4}$ n even, equality iff $n=2^m$

$I(n) \geq \frac{n(n-1)(n-3)}{4}$ n odd, equality iff $n=2^m-1$

More intercalate results

- Heinrich and Wallis also found

$$I(2^{\alpha}+1) \geq 2^{\alpha}(2^{\alpha}(2^{\alpha}-5)/4+2)$$

- Kotzig and Zaks (1983) found

$$I(4k+1) \leq 2k(8k^{2}-4k-1), k \geq 1$$

Donovan's lcs result

- Donovan (1998) found:

$$lcs(2m) \geq \frac{5m^2 - 3m}{2}$$

by using the dihedral group table.

Stinson and van Rees's lcs result

- Stinson and van Rees (1982) found:

$$lcs(2^n) \geq 4^n - 3^n$$

by using the elementary abelian 2-group table.

Fu, Fu and Liao's lcs result

- Fu, Fu, and Liao (1995) found:

$$lcs(2^n - 1) \geq 4^n - 3^n - 2^{n+1} + 3$$

by using the elementary abelian 2-group table, minus a row and column.

Gower's lcs result

- Gower (2000) found:

$$lcs(3^n) \geq 9^n - 7^n$$

by using the elementary abelian 3-group table.

The first lcs result

- Nelder (1979) gave a construction leading to:

$$lcs(n) \geq \frac{n^2 - n}{2}$$

Every other known lcs construction in the literature is based upon a Latin square construction which maximizes the number of 2x2 or 3x3 subsquares.

Obvious deduction

- If we are looking for new constructions to maximize lcs, we should try to find a construction for an n x n Latin square which maximizes the number of subsquares.
- Nelder, 1977:

"The setter knows of no general solution, and suggests that solutions be sought first for n prime, then for n a power of a prime, then for general n. He has conjectured the answer for n prime."

- First, check if Heinrich and Wallis's results can be improved; second, try to maximize the number of m x m subsquares, where m>3.

Can Heinrich and Wallis's results be improved?

0	2	1	3	4	5	6	7	8	9	10	11
1	0	2	5	3	4	8	6	7	10	11	9
2	1	0	4	5	3	7	8	6	11	9	10
3	5	4	0	1	2	9	10	11	6	7	8
4	3	5	2	0	1	10	11	9	8	6	7
5	4	3	1	2	0	11	9	10	7	8	6
6	8	7	9	10	11	0	1	2	3	4	5
7	6	8	10	11	9	2	0	1	5	3	4
8	7	6	11	9	10	1	2	0	4	5	3
9	10	11	6	8	7	3	5	4	0	2	1
10	11	9	7	6	8	4	3	2	1	0	2
11	9	10	8	7	6	5	4	3	2	1	0

Construction to increase I(4m)

A^0	$(A^1)^T$	$(A^2)^T$	B^3
A^1	$(A^0)^T$	B^3	$(A^2)^T$
A^2	B^3	$(A^0)^T$	$(A^1)^T$
B^3	A^2	A^1	A^0

A = i-j

B = i+j

We combine 4 dihedral groups to reach a construction which gives

$$I(4m) \geq I(4)m^3$$

Deriving an lcs(4m) result

A^0	$H((A^1)^T)$	$H((A^2)^T)$	$G(B^3)$
$J(A^1)$	$(A^0)^T$	$G(B^3)$	$(A^2)^T$
$J(A^2)$	$G(B^3)$	$(A^0)^T$	$(A^1)^T$
$G(B^3)$	A^2	A^1	A^0

$$lcs(4m) \geq \frac{23m^2 - 9m}{2}$$

$$G(L) = \{(i, j; L_{ij}) \mid (0 \leq i, j \leq m-1) \wedge (m \leq i + j \leq 2m-2)\}$$

$$H(L) = \{(i, j; L_{ij}) \mid 1 \leq j \leq i \leq m-1\} \quad J(L) = \{(i, j; L_{ij}) \mid 1 \leq i \leq j \leq m-1\}$$