Critical Set Problems in Latin Squares

- Empirical problems

- Theoretical problems

What is a *Latin square*?

1	2	3	4	6	5
6	3	1	5	4	2
5	1	4	2	3	6
2	4	5	6	1	3
3	6	2	1	5	4
4	5	6	3	2	1

A *Latin square* is an $n \times n$ array of numbers such that each number in the range $\{$ 1, ..., n $\}$ occurs exactly once in each row and column.

We can also write a Latin square L as a set of ordered triples:

$L = \{(i, j; k)$ | element k occurs in position $(i, j)\}$.

Partial Latin Squares and Unique Completion

A partial Latin square is an $n \times n$ array of numbers such that each number in the range $\{$ 1, ..., n $\}$ occurs at most once in each row and column. It doesn't have to be a subset of a Latin square.

A partial Latin square is said to be have *unique completion* if it is the subset of exactly one Latin square.

Critical sets

A partial Latin square C is said to be a $critical\ set$ if it has unique completion and no smaller subset of C has unique completion.

1	.	.
.	3	.
.	.	.

1	2	3
2	3	1
3	1	2

Latin interchanges

Let P, P' be partial Latin squares.

- *size* $\mid P \mid$

- *shape* $\{(i,j) \mid (i,j;k) \in P, \exists k \in \{1,...,n\}\}$

Assume P, P' have same order, size and shape.

- *mutually balanced* if entries in each row and column of P are same as those in corresponding row and column of P'.

- *disjoint* if no position in P' contains the same entry as the corresponding position of P.

A *latin interchange* I is a partial Latin square for which there exists another partial latin square I', of the same order, size and shape with the property that I and I' are *disjoint* and *mutually balanced*.

1	2	.	.
2	1	.	.
.	.	.	.
.	.	.	.

2	1	.	.
1	2	.	.
.	.	.	.
.	.	.	.

Relationship between Latin interchanges and critical sets

Let L be a Latin square. Then C is a critical set for the Latin square L if and only if:

- $\forall$ interchanges $I \subseteq L, \mid I \cap C \mid \geq 1$.
- $\forall x \in C, \exists$ an interchange $I \subseteq L$ such that $I \cap C = \{x\}$.

How are interchanges found? (I)

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

How is an interchange of size 8 for this Latin square found?

- Take all 4x4 subsquares
- Take all subsets of 8 cells within each 4x4 subsquare
- Permute each row and test if it is still mutually balanced

How are interchanges found? (II)

Take any Latin square L. Since a Latin interchange of size m for L must have at least two entries in each row and column, it has at most $\lfloor \frac{m}{2} \rfloor$ rows and columns.

To find a Latin interchange of size m:

- examine all subsets of size m from all $\lfloor \frac{m}{2} \rfloor \times \lfloor \frac{m}{2} \rfloor$ subsquares

- test to see if they are interchanges

- for each row, each permutation where all the elements differ in value from the original row is examined, and if the column elements are mutually balanced, a Latin interchange has been found.

The $\frac{n^2}{4}$ construction

Let L be a Latin square of order n. L is said to be $back-circulant$ if every position (i,j) contains the element $i+j-1$ (mod n). Then a critical set C of size $\lfloor\frac{n^2}{4}\rfloor$ can be found for L:

$$\begin{aligned} C &= \{(i,j;i+j-1(mod\ n)) \mid \\ &\quad 1 \le i \le \frac{n}{2}-1, 1 \le j \le \frac{n}{2}-1-i\} \\ &\cup \{(i,j;i+j-1(mod\ n)) \mid \\ &\quad \frac{n}{2}+1 \le i \le n, n+\frac{n}{2}-i \le j \le n\}. \end{aligned}$$

1	2	.	.	.	.
2	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	3
.	.	.	.	3	4
.	.	.	3	4	5

More definitions

Two Latin squares L and M are:

- *isotopic* if rows, columns, or elements can be permuted to change L to M.

- *conjugate* if rows, columns, or elements can be interchanged to change L to M.

- in the same *main class* if L and M are either isotopic or conjugate to each other.

Critical sets for order 6 Latin squares

Aim: To try to find a general construction for the critical set of size $\lfloor \frac{n^2}{4} \rfloor + 1$.

I began by considering order 6 Latin squares. Each critical set of size 10 was compared with each critical set of size 9. No pattern was found; the closest two sets are presented below.

Of the twelve main classes of Latin squares of order six, five have a minimal critical set size of 10. The back-circulant Latin square has a minimal critical set size of 9; there are 72 such critical sets. The back-circulant square has no critical sets of size 10.

1	.	.	.	.	6
2	.	.	.	6	1
.	4	5	.	.	.
.	5	.	.	.	.
.	.	.	.	.	.
6	.	.	.	.	.

1	.	.	.	.	6
2	.	4	.	6	.
.	4	5	.	.	.
.	5	.	.	.	.
.	.	.	.	.	.
6	.	.	.	2	.

Critical sets for order 8 Latin squares

There are 283,657 main classes of order 8 Latin squares. (Brendan McKay)

To date, no critical set of order 17 is known for any Latin square of order 8.

$$\binom{64}{17} = 1{,}379{,}370{,}175{,}283{,}520$$

Special classes of 8x8 Latin squares

- Those which split up into 4x4 subsquares

- Those which have exactly 16 disjoint intercalates

1	6	2	4	3	7	5	8
6	1	4	2	7	3	8	5
2	5	3	1	4	8	7	6
5	2	1	3	8	4	6	7
3	8	5	7	6	2	4	1
8	3	7	5	2	6	1	4
7	4	6	8	1	5	3	2
4	7	8	6	5	1	2	3

- Those from which exactly 16 disjoint intercalates can be chosen

5	1	2	4	6	8	3	7
1	5	4	3	8	6	7	2
4	2	5	1	7	3	8	6
3	4	1	5	2	7	6	8
8	7	6	2	5	4	1	3
7	8	3	6	4	5	2	1
6	3	8	7	1	2	5	4
2	6	7	8	3	1	4	5

8x8 Latin squares with 4x4 subsquares (I)

If an 8x8 Latin square L splits up into 4x4 subsquares A, B, C, and D, then any critical set C of size 17 must have 4 elements in three of A, B, C, and D and 5 elements in the other. This is because C must intersect every interchange in A, B, C, and D and thus the part of the critical set that is in the one of these Latin squares must be a critical set for that Latin square.

1	2	5	6	3	4	7	8
2	1	6	5	4	3	8	7
5	6	2	1	7	8	4	3
6	5	1	2	8	7	3	4
3	4	7	8	5	6	2	1
4	3	8	7	6	5	1	2
7	8	4	3	2	1	6	5
8	7	3	4	1	2	5	6

8x8 Latin squares with 4x4 subsquares (II)

Two main classes of 4x4 Latin squares

- back-circulant - 32 critical sets of size 4, 576 of size 5

- non-back-circulant - 96 critical sets of size 5

$32^3 \times 96 = 3,145,728$
$32^3 \times 576 = 18,874,368$

98 candidates.

8x8 Latin squares with exactly 16 disjoint intercalates

If such a square has exactly 16 disjoint intercalates, then any critical set of size 17 must intersect each of these intercalates in one or more positions. We can consider choosing one element from each intercalate and another from the remaining 48.

$$4^{16} \times 48 = 206,158,430,208$$

This can be halved since all the potential critical sets are being counted twice.

8x8 Latin squares from which exactly 16 disjoint intercalates can be chosen

The Latin squares for which more than 16 intercalates exist can be examined and tested to see if the intercalates cover the whole square. A recursive algorithm can be used to attempt to choose 16 disjoint intercalates from the intercalates.

5	1	2	4	6	8	3	7
1	5	4	3	8	6	7	2
4	2	5	1	7	3	8	6
3	4	1	5	2	7	6	8
8	7	6	2	5	4	1	3
7	8	3	6	4	5	2	1
6	3	8	7	1	2	5	4
2	6	7	8	3	1	4	5

1	1	3	3	2	2	2	2
1	1	3	3	3	3	3	3
2	3	2	4	4	4	4	3
3	2	4	2	4	4	3	4
2	2	3	3	4	4	5	5
4	4	5	5	3	3	2	2
3	3	4	4	3	3	4	4
2	2	2	2	5	5	5	5

Parallelisation of the search

The search is run on the supercomputer $ozone$ in the Prentice centre. This has 64 processors and thus for maximum efficiency, the search needs to be split up and run in parallel. This is done simply by keeping a count of the number of critical sets examined.

Direct Products of Latin squares

Let M, N be Latin squares.

Then $M \times N =$
$\{((i_m, i_n), (j_m, j_n); (k_m, k_n)) \mid (i_m, j_m; k_m) \in M \wedge (i_n, j_n; k_n) \in N\}$.

For example, the product of the back-circulant Latin squares of sizes 2 and 5 is:

1	2	3	4	5	6	7	8	9	10
2	3	4	5	1	7	8	9	10	6
3	4	5	1	2	8	9	10	6	7
4	5	1	2	3	9	10	6	7	8
5	1	2	3	4	10	6	7	8	9
6	7	8	9	10	1	2	3	4	5
7	8	9	10	6	2	3	4	5	1
8	9	10	6	7	3	4	5	1	2
9	10	6	7	8	4	5	1	2	3
10	6	7	8	9	5	1	2	3	4

Critical Sets in Direct Products of Latin Squares

1	2	3	4	.	6	7	8	9	.
2	3	4	.	.	7	8	9	.	.
3	4	.	.	.	8	9	.	.	.
4	.	.	.	.	9	.	.	.	.
.	.	.	.	.	.	.	.	.	.
6	7	8	9	.	.	.	.	.	5
7	8	9	.	.	.	.	.	5	1
8	9	.	.	.	.	.	5	1	2
9	.	.	.	.	.	5	1	2	3
.	.	.	.	.	5	1	2	3	4