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k–homogeneous latin trades

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Abstract

Let T be a partial latin square and L a latin square such that $T \subset L$. Then T is called a *latin trade*, if there exists a partial latin square T^* such that $T^* \cap T = \phi$ and $(L \setminus T) \cup T^*$ is a latin square. We call T^* a *disjoint mate* of T. A latin trade is called *k–homogeneous* if each row and each column contains exactly k elements, and each element appears exactly k times. The number of elements in a latin trade is referred to as its *volume*.

It is shown by Cavenagh, Donovan, and Drápal (2003 and 2004) that 3–homogeneous and 4–homogeneous latin trades of volume $3m$ and $4m$, respectively, exist for all $m \geq 3$ and $m \geq 4$, respectively. We show that k–homogeneous latin trades of volume km exist for all $3 \leq k \leq 8$ and $m \geq k$. Also we show that for each given $k \geq 3$ and for $m \geq k$, all k–homogeneous latin trades of volume km exist except possibly for finitely many m, i.e. $k < m < 2k + 20$.

1 Introduction

A *latin square* L of order n is an $n \times n$ array of the set, say $N = \{0, 1, 2, \ldots, n-1\}$, where each element of N appears exactly once in each row and exactly once in each column. We can represent each latin square as a set of 3-tuples

$$L = \{(i, j; k) \mid \text{element } k \text{ is located in position } (i, j)\}.$$

A *transversal* of a latin square of order n is a set of n positions, no two in the same row or same column, containing each elements of the set N exactly once.

A *partial latin square* P of order n is an $n \times n$ array of elements from the set N, where each element of N appears at most once in each row and at most once in each column. The set $S_P = \{(i,j)|(i,j;k) \in P\}$ of the partial latin square P is called the *shape* of P and $|S_P|$ is called the $volume$ of P.
For each $0 \leq r, c, e \leq n-1$ we introduce the following sets:
$C_P^c = \{k|(i,c;k) \in P\}$, $R_P^r = \{k|(r,j;k) \in P\}$ and $E_P^e = \{(i,j)|(i,j;e) \in P\}$.
We call a partial latin square T of order n a *latin trade* if there exists a partial latin square T^* of order n , called a mate of T, such that:
$S_T = S_{T^*}$, and if $(i,j;k) \in T$ and $(i,j;k^*) \in T^*$, then $k \neq k^*$,
for each r, $0 \leq r \leq n-1$, we have $R_T^r = R_{T*}^r$; and
for each c, $0 \leq c \leq n-1$, we have $C_T^c = C_{T*}^c$.
A latin trade is called k-*homogeneous* if:
for each r, $0 \leq r \leq n-1$, we have $|R_T^r| = 0$ or k;
for each c, $0 \leq c \leq n-1$, we have $|C_T^c| = 0$ or k; and
for each e, $0 \leq e \leq n-1$, we have $|E_T^e| = 0$ or k.
A latin trade of volume 4 is called an *intercalate*. In Figure 1 an intercalate (T, T^*) is shown. The elements of T^* is written as subscripts in the same array as T.

0_1	1_0
1_0	0_1

Figure 1: An intercalate

For more background on latin trades see [2], [6], and [5], and concepts which are not defined here may be found in [1]. It is proved in [4] and [3], that 3-homogeneous Latin trades of volume $3m$ exist for all $m \geq 3$, and 4-homogeneous latin trades of volume $4m$ exist for all $m \geq 4$. We prove that for each given $k \geq 3$ and for $m \geq k$, all k–homogeneous latin trades of volume km exist except possibly for finitely many m. We also show that for $3 \leq k \leq 8$ and $m \geq k$, k–homogeneous latin trades of volume km exist. It is obvious that we can omit empty rows and columns of a latin trade. Hence without loss of generality, we can assume any k–homogeneous latin trade of volume km is located in an $m \times m$ square.

2 Results

For each k, there exists at least a k–homogeneous latin trade of volume k^2. To see this, for a latin square L of order k, we can take L^* to be a latin square with a cyclic permutation on the rows of L. So L^* is a disjoint mate of L.

Theorem 1 *If $l \neq 2, 6$ and for each $k \in \{k_1, \ldots, k_l\}$ there exists a k–homogeneous latin trade of volume kp, then a $(k_1 + \cdots + k_l)$–homogeneous latin trade of volume $(k_1 + \cdots + k_l)lp$ exists. (Some k_is can possibly be zero).*

Proof. Since $l \neq 2, 6$, there exist two $l \times l$ orthogonal latin squares. Denote one of them by L and partition L into l transversals. Consider disjoint sets A_1, A_2, ..., A_l, each with p elements. Then, if $k_i \neq 0$ replace the element j in the i–th transversal of L by a k_i–homogeneous latin trade of volume $k_i p$ with elements taken from A_j, and if $k_i = 0$, then we replace each element of i–th transversal by an empty $p \times p$ square. ■

A latin square is *self-orthogonal* if it is orthogonal to its transpose. A self-orthogonal latin square (SOLS) of order n will be denoted by SOLS(n).

Theorem 2 *For each $k > 2$, a k–homogeneous latin trade of volume $k(k + 1)$ exists.*

Proof. By Theorem 6.1.3 of [1],on page 139, we know that SOLS(n) exists for every $n \neq 2, 3$, and 6. By deleting the main diagonals in an SOLS(n) and in its transpose, we obtain $(n - 1)$–homogeneous latin trade of volume $n(n - 1)$ with its disjoint mate. An example of a 5–homogeneous latin trade of volume 30 is shown in Figure 2.

·	3_2	2_3	0_4	4_5	5_0
5_2	4_1	0_4	·	1_0	2_5
4_3	1_4	·	2_0	3_1	0_2
0_4	5_3	4_0	1_5	·	3_1
2_5	·	3_1	4_2	5_4	1_3
3_0	2_5	1_2	5_1	0_3	·

Figure 2: A 5–homogeneous latin trade of volume 30.

And this example completes the proof. ■

Theorem 3 *Any 2-homogeneous latin trade can be partitioned into disjoint intercalates.*

Proof. We prove by induction. Suppose T is a 2-homogeneous latin trade of volume $2m$. Without loss of generality, we may let $\{(0, 0; 0), (0, 1; 1), (1, 0; 1)\} \subseteq T$. Then T must contain $(1, 1; 0)$. We can apply the same argument to the $(m - 2) \times (m - 2)$ subsquare obtained by removing rows 0 and 1, and columns 0 and 1. This completes the proof. ■

Theorem 4 *For every k, if there exists a k–homogeneous latin trade of volume km and a k–homogeneous latin trade of volume kn, then for each r and $s \geq 0$, there exists a k–homogeneous latin trade of volume $k(rm + sn)$.*

Proof. Let $T_1, \ldots, T_r$ be k–homogeneous latin trades of volume km and $T_{r+1}, \ldots, T_{r+s}$ be k–homogeneous latin trades of volume kn such that for each i, $1 \leq i \leq r$ elements of T_i are in the set $\{(i-1)m+1, \ldots, im\}$ and for each j, $1 \leq j \leq s$ elements of T_{j+r} are in the set $\{rm + (j-1)n + 1, \ldots, rm + jn\}$. The following latin trade is a k–homogeneous latin trade of volume $k(rm + sn)$.

$$\begin{array}{|cccc|}\hline T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_{r+s} \\ \hline \end{array}$$

■

Corollary 5 *For each k and m where $m \geq k^2$, there exists a k–homogeneous latin trade of volume km.*

Proof. If $m \geq k^2$, then we can write m as $m = rk + s(k+1)$, where $r, s \geq 0$. Theorem 4 and Theorem 2 lead us to conclusion. ■

Theorem 6 *If $(m, k) = d$, $m \geq k$, and $d > 1$, then there exists a k–homogeneous latin trade of volume km.*

Proof. For $m = k$, the theorem is trivial. Now suppose that $m \neq k$, so $\frac{m}{d} \geq 2$. Let $m' = \frac{m}{d}$ and $k' = \frac{k}{d}$. We construct a k–homogeneous latin trade of volume km in the following way. Consider an $m' \times m'$ latin square L on the set $\{1, 2, \ldots, m'\}$. We replace each i in L with,

- a d–homogeneous latin trade of volume d^2 whose elements are from the set $\{(i-1)d + 1, \ldots, id\}$, if $1 \leq i \leq k'$; and
- an empty $d \times d$ array, if $k' + 1 \leq i \leq m'$.

So we obtain a k–homogeneous latin trade of volume km. Note that its mate can be obtained by replacing each d–homogeneous latin trade of volume d^2 with its mate. In Figure 3, we have an example of the case $k = 3d$ and $m = 5d$.

T_1	T_2	T_3	•	•
•	T_1	T_2	T_3	•
•	•	T_1	T_2	T_3
T_3	•	•	T_1	T_2
T_2	T_3	•	•	T_1

T_1^*	T_2^*	T_3^*	•	•
•	T_1^*	T_2^*	T_3^*	•
•	•	T_1^*	T_2^*	T_3^*
T_3^*	•	•	T_1^*	T_2^*
T_2^*	T_3^*	•	•	T_1^*

■

Figure 3: A latin trade is constructed for $k^{'} = 3$ and $m^{'} = 5$

By using Theorem 6 and Theorem 3, we have the following corollary:

Corollary 7 *For any $m \geq 1$, there exists a 2–homogeneous latin trade of volume $2m$, if and only if m is an even number.*

Theorem 8 *For any $m = 4l$ and $2 \leq k \leq m$, there exists a k–homogeneous latin trade of volume km.*

Proof. It is easy to see that theorem holds for $l = 1$. Also we may assume that $m > k$.
First, we prove the theorem in the case that $l \neq 2, 6$. We have the following cases to consider:

1. k is even. This case follows from Theorem 6.
2. $k = 4l^{'} + 1$. This case follows from Theorem 1. Indeed, we set $p = 4$ and $k_i = 4$ for $1 \leq i \leq l^{'} - 1$, $k_{l'} = 2$, $k_{l'+1} = 3$, and $k_i = 0$ otherwise.
3. $k = 4l^{'} + 3$. This case also follows from Theorem 1, by setting $p = 4$ and $k_i = 4$ for $1 \leq i \leq l^{'}$, $k_{l'+1} = 3$, and $k_i = 0$ otherwise.

Next, we study the cases $l = 2$ and $l = 6$. Actually for $m = 8$ or 24, the statement follows from Theorem 1, Theorem 2, Theorem 4 and Theorem 6, by choosing proper k_is and p, except the case $m = 8$ and $k = 5$. The existence of the last case is shown in Appendix. ∎

Theorem 9 *For any $m = 5l$ and $3 \leq k \leq m$, there exists a k–homogeneous latin trade of volume km.*

Proof. Theorem trivially holds for $l = 1$. We may also assume that $m > k$.
First, we prove the theorem in the case that $l \neq 2 \text{ and } 6$. We have the following cases to consider:

1. $k = 5l^{'}$. Obviously this case follows from Theorem 6.
2. $k = 5l^{'} + 1$. This case follows from Theorem 1. Indeed, we set $k_i = 5$ for $1 \leq i \leq l^{'} - 1$ and $k_{l'+1} = k_{l'} = 3$ and $k_i = 0$ for $l^{'} + 2 \leq i \leq l$ and $p = 5$.
3. $k = 5l^{'} + 2$. This case follows from Theorem 1, if we set $k_i = 5$ for $1 \leq i \leq l^{'} - 1$ and $k_{l'} = 3$ and $k_{l'+1} = 4$ and $k_i = 0$ for $l^{'} + 2 \leq i \leq l$ and $p = 5$.

4. $k = 5l^{'} + r$, $r = 3, 4$. This case also follows from Theorem 1, if we set $k_i = 5$ for $1 \leq i \leq l'$, $k_{l'+1} = r$, and $k_i = 0$ for $l^{'} + 2 \leq i \leq l$, and $p = 5$.

Next, for the case $l = 2$, by Theorem 1, Theorem 2, Theorem 4, and Theorem 6, we can show that there exists a k–homogeneous latin trade of volume $10k$ for any $3 \leq k \leq 10$ and $k \neq 7$. For $k = 7$, a 7–homogeneous latin trade of volume 70 is shown in Appendix.
If $l = 6$, by using Theorem 1, Theorem 2, Theorem 4 and Theorem 6, we can construct a k–homogeneous latin trade of volume $30k$ for any $3 \leq k \leq 30$, by using k_i–homogeneous latin trades of volume $6k_i$, where $k_1 + k_2 + k_3 + k_4 + k_5 = k$, $0 \leq k_i \leq 6$ and $k_i \neq 1$ for any i, $1 \leq i \leq 5$. ■

Theorem 10 *For any $k \geq 3$ and $m \geq 2k + 20$, there exists a k–homogeneous latin trade of volume km.*

Proof. Consider an arbitrary $m \geq 26$. We can represent it as $m = 4r + 5s$, where r and s are positive integers. It is not hard to see that $s \equiv m \pmod 4$ and $r \equiv 4m \pmod 5$. So we can conclude that, there exist unique $0 \leq r^{'} \leq 4$ and $0 \leq s^{'} \leq 3$, such that $r = 5a + r^{'}$ and $s = 4b + s^{'}$, where $a, b \geq 0$. It yields that $m = 4r + 5s = 4(5a + r^{'}) + 5(4b + s^{'})$. We conclude that $a + b = \frac{m - 4r^{'} - 5s^{'}}{20}$ is a constant number. Now we have two following cases:

- $a + b$ is even. In this case set $a = b$.
- $a + b = 2t + 1$. If $5s^{'} > 4r^{'}$, set $a = t + 1$ and $b = t$, otherwise set $a = t$ and $b = t + 1$.

In each of these cases, we have $|4r - 5s| \leq 20$. And we have $m = 4r + 5s$, where $4r, 5s \geq m/2 - 10$. By Theorem 8 and Theorem 9, for any $3 \leq k \leq m/2 - 10$ we have k–homogeneous latin trades of volume $4kr$ and $5ks$. Now by Theorem 4, we conclude that there exists a k–homogeneous latin trade of volume $4kr + 5ks$. ■

The following theorem results immediately from Theorem 10.

Main Theorem 1 *For each given $k \geq 3$, and for $m \geq k$, all k–homogeneous latin trades of volume km exist except possibly for finitely many m.*

Theorem 11 *Consider an arbitrary natural number k. If for any $k + 1 \leq m \leq 2k - 1$ there exists a k–homogeneous latin trade of volume km, then for any $m \geq k$ there exists a k–homogeneous latin trade of volume km.*

Proof. For any $m \geq 2k$, we can write $m = rk + sl$, where $r, s \geq 0$ and $k + 1 \leq l \leq 2k - 1$. Since there exist k–homogeneous latin trades of volume k^2 and kl, by Theorem 4 we conclude that there exists a k–homogeneous latin trade of volume km. ■

Main Theorem 2 *For any $3 \leq k \leq 8$ and $m \geq k$, there exists a k–homogeneous latin trade of volume km.*

Proof. For $k = 3$ or $k = 4$ the theorem is proved in [4] and [3], respectively. By Theorem 11, we only need to show the existence of a k–homogeneous latin trade of volume km, for each $5 \leq k \leq 8$ and $k + 1 \leq m \leq 2k - 1$. If $m = k + 1$ statement follows from Theorem 2. For $m > k + 1$, we consider the following four cases:

- **Case 1.** $k = 5$.

$m = 7$. An example of a 5–homogeneous latin trade of volume 35 is given in Appendix.

$m = 8$. It follows from Theorem 8.

$m = 9$. It follows from Theorem 1, by setting $k_1 = 0$, $k_2 = 2$, $k_3 = 3$ and $p = 3$.

- **Case 2.** $k = 6$.

$m = 8, 9$ or 10. All follow from Theorem 6.

$m = 11$. An example of a 6–homogeneous latin trade of volume 66 is given in Appendix.

- **Case 3.** $k = 7$.

$m = 9$. It follows from Theorem 1, by setting $k_1 = 2$, $k_2 = 2$, $k_3 = 3$ and $p = 3$.

$m = 10$. It follows from Theorem 9.

$m = 12$. It follows from theorem 8.

$m = 11$ or $m = 13$. Examples of 7–homogeneous latin trades of volume 77 and 91 is given in Appendix.

- **Case 4.** $k = 8$.

$m = 10$, $m = 12$ or $m = 14$. It follows from Theorem 6.

$m = 15$. It follows from Theorem 9.

$m = 11$ or $m = 13$. Examples of 8–homogeneous latin trades of volume 88 and 104 are given in Appendix.

■

3 Future Research

For future research, we could look at ensuring the latin trades we create are minimal. When we wish to determine if a set of entries is a subset of exactly one latin square, it is faster to use only minimal latin trades. Thus, we can look at the problem of whether minimal k-homogeneous latin trades of volume km exist for all k and m.

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4 Appendix

In the following all required k–homogeneous latin trades for the mentioned results are given.

5–homogeneous latin trades of volume 35 and 40:

2_0	4_2	0_1	3_4	1_3	·	·
3_2	5_1	2_0	0_5	·	1_3	·
·	·	4_2	5_6	2_5	3_4	6_3
·	2_5	1_6	6_3	·	5_1	3_2
6_3	1_6	·	4_0	3_4	·	0_1
5_6	·	6_4	·	4_2	0_5	2_0
0_5	6_4	·	·	5_1	4_0	1_6

7_0	0_1	3_2	2_3	·	·	·	1_7
0_1	1_0	4_3	3_2	·	2_4	·	·
·	·	·	7_1	1_6	6_7	5_4	4_5
·	4_2	·	·	6_7	7_6	2_5	5_4
·	2_5	5_6	6_7	7_0	·	0_2	·
6_5	5_4	·	1_6	3_1	·	4_3	·
1_6	·	6_4	·	·	4_3	3_0	0_1
5_7	·	2_5	·	0_3	3_2	·	7_0

A 6–homogeneous latin trade of volume 66 and a 7–homogeneous latin trade of volume 70:

·	·	6_2	·	·	7_5	5_6	a_7	2_8	·	8_a
9_1	·	·	7_4	·	·	a_7	1_8	8_9	4_a	·
·	6_3	·	0_5	3_6	8_7	7_8	·	·	5_0	·
8_3	·	2_5	·	·	1_8	·	5_a	·	a_1	3_2
1_4	·	7_6	4_7	·	·	6_a	·	3_1	·	a_3
·	7_6	9_7	·	1_9	·	·	2_1	4_2	·	6_4
·	3_7	·	2_9	·	4_0	·	7_2	9_3	0_4	·
·	·	0_9	·	9_0	5_1	·	·	1_4	6_5	4_6
3_8	0_9	·	9_0	6_1	·	8_3	·	·	1_6	·
a_9	9_a	5_0	·	0_2	·	·	8_5	·	·	2_8
4_a	a_0	·	5_2	2_3	0_4	3_5	·	·	·	·

·	·	·	3_2	1_3	5_9	2_5	7_1	8_7	9_8
·	·	·	7_3	5_4	3_5	4_6	8_7	9_8	6_9
·	5_2	·	·	7_5	0_1	1_7	9_8	2_9	8_0
1_2	9_3	5_4	0_5	·	·	·	3_9	4_0	2_1
2_3	3_4	7_5	5_6	0_7	·	·	6_0	·	4_2
5_9	4_5	9_1	2_7	·	6_4	·	1_6	7_2	·
7_5	8_6	0_7	6_8	·	4_0	5_1	·	·	1_4
8_1	·	4_8	·	2_0	1_6	3_2	0_3	6_4	·
9_7	6_8	1_9	8_0	3_1	·	7_3	·	·	0_6
3_8	2_9	8_0	·	4_2	9_3	6_4	·	0_6	·

A 7–homogeneous latin trades of volume 77:

·	·	·	·	6_4	4_5	9_6	8_7	a_8	5_9	7_a
·	·	·	8_4	·	7_6	6_7	a_8	0_9	9_a	4_0
·	·	·	7_5	0_6	6_7	5_8	·	8_a	1_0	a_1
5_3	9_4	0_5	·	·	·	3_9	1_a	4_0	a_1	·
6_4	7_5	5_6	4_7	·	·	·	2_0	·	3_2	0_3
4_5	5_6	·	1_8	·	·	·	6_1	3_2	2_3	8_4
8_6	6_7	a_8	2_9	3_a	·	·	7_2	9_3	·	·
9_7	·	1_9	·	4_0	5_1	7_2	·	2_4	0_5	·
7_8	a_9	9_a	·	2_1	1_2	8_3	·	·	·	3_7
3_9	0_a	8_0	9_1	a_2	2_3	·	·	·	·	1_8
·	4_0	6_1	5_2	1_3	3_4	2_5	0_6	·	·	·

A 7–homogeneous latin trade of volume 91:

·	·	·	·	9_4	·	7_6	4_7	a_8	c_9	6_a	·	8_c
·	·	·	9_4	·	0_6	6_7	·	7_9	b_a	a_b	·	4_0
·	·	·	·	a_6	·	·	6_9	b_a	9_b	0_c	1_0	c_1
5_3	·	c_5	·	·	·	·	·	2_b	3_c	b_0	0_1	1_2
·	6_5	1_6	b_7	·	·	·	7_b	·	·	5_1	3_2	2_3
9_5	0_6	·	4_8	6_9	·	·	·	8_0	·	·	5_3	3_4
·	b_7	9_8	a_9	4_a	2_b	·	·	·	7_2	·	8_4	·
3_7	7_8	5_9	8_a	·	·	·	·	9_2	a_3	·	2_5	·
a_8	·	8_a	1_b	0_c	b_0	5_1	·	·	·	c_5	·	·
8_9	·	·	·	2_0	6_1	3_2	9_3	·	·	1_6	·	0_8
c_a	8_b	a_c	·	·	3_2	2_3	b_4	·	·	·	4_8	·
·	5_c	·	7_1	c_2	4_3	1_4	3_5	·	2_7	·	·	·
7_c	c_0	6_1	·	·	1_4	4_5	5_6	0_7	·	·	·	·

An 8–homogeneous latin trades of volume 88:

·	·	·	4_3	3_4	6_5	5_6	8_7	9_8	a_9	7_a
·	·	·	5_4	0_5	4_6	6_7	7_8	8_9	9_a	a_0
·	·	·	7_5	1_6	5_7	9_8	a_9	0_a	6_0	8_1
4_3	0_4	a_5	·	·	·	2_9	9_a	1_0	5_1	3_2
5_4	6_5	7_6	2_7	·	·	·	1_0	3_1	0_2	4_3
7_5	5_6	8_7	3_8	·	·	·	6_1	4_2	2_3	1_4
3_6	9_7	6_8	8_9	2_a	7_0	·	0_2	a_3	·	·
9_7	7_8	5_9	·	4_0	0_1	8_2	·	2_4	1_5	·
a_8	8_9	9_a	·	6_1	1_2	7_3	·	·	3_6	2_7
8_9	4_a	1_0	9_1	a_2	2_3	3_4	·	·	·	0_8
6_a	a_0	0_1	1_2	5_3	3_4	4_5	2_6	·	·	·

An 8–homogeneous latin trades of volume 104:

·	·	·	·	·	b_5	5_6	8_7	7_8	a_9	6_a	c_b	9_c
·	·	·	b_4	·	·	8_7	9_8	4_9	7_a	a_b	0_c	c_0
·	·	·	·	a_6	·	9_8	6_9	b_a	c_b	0_c	1_0	8_1
c_3	·	b_5	·	·	·	3_9	·	2_b	9_c	1_0	5_1	0_2
5_4	7_5	0_6	1_7	·	·	·	·	·	6_0	2_1	3_2	4_3
8_5	9_6	6_7	7_8	3_9	·	·	·	·	·	5_2	4_3	2_4
4_6	8_7	7_8	a_9	6_a	1_b	·	·	9_1	·	·	b_4	·
3_7	c_8	·	8_a	·	5_c	7_0	·	a_2	0_3	·	2_5	·
a_8	b_9	8_a	9_b	0_c	2_0	·	5_2	·	·	c_5	·	·
·	6_a	a_b	·	1_0	0_1	·	4_3	8_4	·	b_6	·	3_8
6_a	a_b	·	·	9_1	3_2	4_3	2_4	·	b_6	·	·	1_9
·	5_c	1_0	2_1	c_2	4_3	0_4	7_5	·	3_7	·	·	·
7_c	·	5_1	4_2	2_3	c_4	6_5	3_6	1_7	·	·	·	·