Disjoint critical sets in Latin squares

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ABSTRACT: A critical set in a Latin square is a subset of the Latin square containing just enough information to determine the complete Latin square. It has been conjectured that the smallest possible critical set in a Latin square is of size $\lfloor \frac{n^2}{4} \rfloor$. If this conjecture is true, it may be possible to partition a Latin square L into two, three or four disjoint critical sets in L. We give a theorem to show that for a given order n, there exists a back-circulant Latin square of order n which may be partitioned into four disjoint critical sets, and we give examples of all possible different partitions of Latin squares of order at most 6. We also give an example of two mutually complementary critical sets, which partition a Latin square of order 8 into two disjoint critical sets.

1 Introduction

A *partial Latin square* P of order n is an $n \times n$ array containing symbols chosen from a set N of size n in such a way that each element of N occurs at most once in each row and at most once in each column of the array. For ease of exposition, a partial Latin square P will be represented by a set of ordered triples $\{(i, j; k) \mid \text{element } k \in N \text{ occurs in cell } (i, j) \text{ of the array}\}$. If all the cells of the array are filled then the partial Latin square is termed a Latin square. That is, a *Latin square* L of order n is an $n \times n$ array with entries chosen from the set N in such a way that each element of N occurs precisely once in each row and precisely once in each column of the array.

A *critical set* in a Latin square L (of order n) is a partial Latin square $\mathcal{C}$ in L, such that

(1) L is the only Latin square of order n which has element k in cell (i, j) for each $(i, j; k) \in \mathcal{C}$, and

(2) no proper subset of $\mathcal{C}$ satisfies (1).

A *uniquely completable set* (UC) in a Latin square L of order n is a partial Latin square in L which satisfies condition (1) above.

Define $\mathrm{scs}(n)$ to be the cardinality of smallest critical sets in any Latin square of order n. Mahmoodian in [10] and Bate and van Rees in [4] independently conjectured that $\mathrm{scs}(n) = \lfloor n^2/4 \rfloor$. This conjecture has been shown to be true for $1 \leq n \leq 7$ (see [2, 4, 6, 8, 9]). If this conjecture turns out to be true then the maximum number of disjoint critical sets in a Latin square of order n will be four.

In this paper, we examine representatives of each main class of Latin square (see [5]) of order up to six in order to determine which main classes can be partitioned into disjoint critical sets. General results are given which partition the back-circulant Latin square of non-trivial order into four disjoint critical sets.

We also consider the problem of partitioning a Latin square into two disjoint critical sets. This is shown to be impossible for Latin squares of order less than or equal to six. Some partial non-existence results are given for the problem for Latin squares of order seven, and the first published example of a solution to this problem is given. It is derived from a Latin square of order eight.

The notation $LS_{y.z}$ denotes main class z in a Latin square of order y, as in the CRC Handbook of Combinatorial Designs, [5]. For the tables in this paper, we shall denote a critical set of size x by $\mathcal{C}(x)$.

2 Theoretical results

Let B_n denote the back circulant Latin square of order $n > 1$. That is, if the rows and columns are labelled zero to $n-1$, then for all positive integers n,

$$B_n = \{(i, j; i + j (\mathrm{mod}\ n)) \mid 0 \leq i, j \leq n-1\}.$$

(Note that here $N = \{0, 1, 2, \ldots, n-1\}$.)

Theorem 2.1 Let B_n be the back circulant Latin square of order n.

(1) If n is even then B_n can be partitioned into four critical sets of size $\frac{n^2}{4}$.

(2) If n is odd then B_n can be partitioned into three critical sets of size $\frac{n^2-1}{4}$ and one critical set of size $\frac{n^2+3}{4}$.

Proof: (1) Define

$$\begin{aligned} \mathcal{C}_1 \quad = \quad & \{(i, j; i + j) \mid i = 0, \ldots, \tfrac{n}{2} - 1 \text{ and } j = 0, \ldots, \tfrac{n}{2} - 1 - i\} \bigcup \\ & \{(i, j; i + j) \mid i = \tfrac{n}{2} + 1, \ldots, n-1 \text{ and } j = \tfrac{n}{2} - i, \ldots, n-1\} \end{aligned}$$

(addition is modulo n). It is well-known (see for example [7]) that $\mathcal{C}_1$ is a critical set of size $\frac{n^2}{4}$ in B_n. Now we define

$$\begin{aligned}
\mathcal{C}_2 &= \{(i, j+\frac{n}{2}; i+j+\frac{n}{2}) \mid (i,j;i+j) \in \mathcal{C}_1\}, \\
\mathcal{C}_3 &= \{(i+\frac{n}{2}, j; i+j+\frac{n}{2}) \mid (i,j;i+j) \in \mathcal{C}_1\}, \text{ and} \\
\mathcal{C}_4 &= \{(i+\frac{n}{2}, j+\frac{n}{2}; i+j) \mid (i,j;i+j) \in \mathcal{C}_1\}.
\end{aligned}$$

It is easy to see that $\mathcal{C}_i$ is also a critical set of size $\frac{n^2}{4}$ in B_n for $i = 2, 3, 4$. Moreover, $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $1 \leq i < j \leq 4$. Finally, $B_n = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$.
(2) Define

$$\begin{aligned}
\mathcal{C} = &\{(i,j;i+j) \mid i = 0, \ldots, \frac{n-1}{2} \text{ and } j = 0, \ldots, \frac{n-1}{2} - i\} \bigcup \\
&\{(i,j;i+j) \mid i = \frac{n-1}{2} + 2, \ldots, n-1 \text{ and } j = \frac{n-1}{2} + 1 - i, \ldots, n-1\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_1 = &\{(i,j;i+j) \mid i = 0, \ldots, \frac{n-3}{2} \text{ and } j = 0, \ldots, \frac{n-3}{2} - i\} \bigcup \\
&\{(i,j;i+j) \mid i = \frac{n-1}{2} + 1, \ldots, n-1 \text{ and } j = \frac{n-1}{2} - i, \ldots, n-1\}.
\end{aligned}$$

It is well-known (see for example [7]) that $\mathcal{C}$ and $\mathcal{C}_1$ are critical sets of sizes $\frac{n^2-1}{4} + 1$ and $\frac{n^2-1}{4}$, respectively, in B_n. Now we define

$$\begin{aligned}
\mathcal{C}_2 &= \{(i, j+\frac{n+1}{2}; i+j+\frac{n+1}{2}) \mid (i,j;i+j) \in \mathcal{C}_1\}, \\
\mathcal{C}_3 &= \{(i+\frac{n+1}{2}, j; i+j+\frac{n+1}{2}) \mid (i,j;i+j) \in \mathcal{C}_1\}, \text{ and} \\
\mathcal{C}_4 &= \{(\frac{n-1}{2} - i, \frac{n-1}{2} - j; n-1-(i+j) \mid (i,j;i+j) \in \mathcal{C}\}.
\end{aligned}$$

It is easy to see that $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are also critical sets of sizes $\frac{n^2-1}{4}$, $\frac{n^2-1}{4}$ and $\frac{n^2-1}{4}+1$, respectively, in B_n. Moreover, $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $1 \leq i < j \leq 4$. Finally, $B_n = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$.

The following examples illustrate Theorem 2.1.

Example 2.2 Let $n = 6$. Then B_6 is decomposed into

0	1	2			
1	2				
2					
					3
				3	4

$\mathcal{C}_1$

			3	4	5
			4	5	
			5		
		0			
	0	1			

$\mathcal{C}_2$

					0
				0	1
3	4	5			
4	5				
5					

$\mathcal{C}_3$

		3			
	3	4			
			0	1	2
			1	2	
			2		

$\mathcal{C}_4$

Note that $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ are critical sets in B_6 all of size 9.

Let $n = 5$. Then B_5 is decomposed into

0	1			
1				
				2
			2	3

$\mathcal{C}_1$

			3	4
			4	
		0		
	0	1		

$\mathcal{C}_2$

				0
			0	1
3	4			
4				

$\mathcal{C}_3$

		2		
	2	3		
2	3	4		
			1	

$\mathcal{C}_4$

Note that $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ are critical sets of size 6 in B_5 and $\mathcal{C}_4$ is a critical set of size 7 in B_5.

Theorem 2.3 *Let L be a Latin square of order n and $2 \le n \le 6$. Then there do not exist two disjoint critical sets $\mathcal{C}_1$ and $\mathcal{C}_2$ in L such that $L = \mathcal{C}_1 \cup \mathcal{C}_2$.*

Proof: Assume $\mathcal{C}_1$ and $\mathcal{C}_2$ are two disjoint critical sets in L such that $L = \mathcal{C}_1 \cup \mathcal{C}_2$. This follows that either $|\mathcal{C}_1| \ge \lceil n^2/2 \rceil$ or $|\mathcal{C}_2| \ge \lceil n^2/2 \rceil$. The largest possible size for a critical set of order n, where $2 \le n \le 6$, is 1, 3, 7, 11 and 18, respectively (see [1]). So the proof completes for $n = 2, 3, 4, 5$. There are precisely 648 critical sets of order 6 and size 18 (see [1]). These critical sets occur in a Latin square which can be partitioned into four subsquares of order 3. There are two simple observations we can make to see that no pair of these critical sets is disjoint. The first is that each of these critical sets has a missing row, column, and element. Thus the complement of any critical set must contain a complete row and column and one element must occur six times. Then no complement can be a critical set. We may also observe that each of these critical sets contains a complete subsquare of order 3. Thus, there is a subsquare missing from each complement and no complement can have unique completion.

We note that the largest critical set known in a Latin square of order 7 is of size 25, and that all known critical sets of this size and order contain a missing row, column, or element (see [3]). Thus no pair of disjoint critical sets of order 7 which cover a complete Latin square is known.

However, for Latin squares of order 8, we have discovered a pair of disjoint critical sets in the abelian 2-group of order 8, denoted $\mathbb{Z}_2^3$, which are mutually complementary. Thus this is the smallest known example. These are shown in the diagram below.

$\mathbb{Z}_2^3 =$

		3	4		6	7	
2	1				5	8	
	4	1				5	6
	3	2		8	7		
		7	8	1			4
6	5			2			3
7			6			1	2
8			5	4	3		

$\mathcal{C}(32)$

$+$

1	2			5			8
		4	3	6			7
3			2	7	8		
4			1			6	5
5	6				2	3	
		8	7		1	4	
	8	5		3	4		
	7	6				2	1

$\mathcal{C}(32)$

3 Computational results

The following results have been obtained through the use of computer searches in Latin squares of orders 4, 5 and 6. We describe the approach used.

Firstly, we generated all possible critical sets for the main classes of Latin squares of order n, $4 \leq n \leq 6$ (see [1]). The possible sizes of these critical sets were noted. Then, all partitions of the number n^2 into these possible sizes were calculated. By considering each of the critical sets as binary strings, and using the binary operations of AND, XOR, and OR, we efficiently calculated which critical sets were disjoint and covered the complete Latin square for each Latin square examined.

These results give existence and non-existence results for all possible sizes of partitions. An existence result is denoted by an equation with a specific example or a reference to Theorem 2.1. A non-existence result, denoted by an inequality, means that no partition with critical sets of the given sizes is possible.

Latin square 4.1 $= LS_{4.1}$

$LS_{4.1} = \mathcal{C}(4) + \mathcal{C}(4) + \mathcal{C}(4) + \mathcal{C}(4)$ (by Theorem 2.1)

$LS_{4.1} =$

	2		
2		4	
		1	
	1		

$\mathcal{C}(5)$

$+$

		3	
	3		1
			2
		2	

$\mathcal{C}(5)$

$+$

1			4
3	4		
4			3

$\mathcal{C}(6)$

Latin square 4.2 $= LS_{4.2}$

$LS_{4.2} =$

1	2		
			3
		1	
4			

$\mathcal{C}(5)$

$+$

			4
	1		
3	4		
		2	

$\mathcal{C}(5)$

$+$

		3	
2		4	
			2
	3		1

$\mathcal{C}(6)$

Latin square 5.1 $= LS_{5.1}$

$LS_{5.1} = \mathcal{C}(6) + \mathcal{C}(6) + \mathcal{C}(6) + \mathcal{C}(7)$ (by Theorem 2.1)

$LS_{5.1} =$

				5
2	3	4		
3	4			
4				

$\mathcal{C}(7)$

$+$

1			4	
			5	1
				2
	5			
5	1	2		

$\mathcal{C}(9)$

$+$

	2	3		
		5	1	
		1	2	3
			3	4

$\mathcal{C}(9)$

$LS_{5.1} =$

1				5
	3	4	5	
3	4			
4				

$\mathcal{C}(8)$

$+$

	2	3	4	
				1
			2	
5	1			4

$\mathcal{C}(8)$

$+$

2				
		5	1	2
	5	1		3
		2	3	

$\mathcal{C}(9)$

Latin square 5.2 $= LS_{5.2}$

$LS_{5.2} =$

	2			5
2				3
		5		
4			3	

$\mathcal{C}(7)$

$+$

		3		
			5	
3			1	2
	5	2		
5			2	

$\mathcal{C}(9)$

$+$

1			4	
	1	4		
	4			
				1
	3	1		4

$\mathcal{C}(9)$

$LS_{5.2} =$

				5
2	1	4		
3	4			
4			3	

$\mathcal{C}(8)$

$+$

1		3		
			5	
		2		1
5	3	1		

$\mathcal{C}(8)$

$+$

	2		4	
				3
		5	1	2
	5			
			2	4

$\mathcal{C}(9)$

Latin square 6.1 $= LS_{6.1}$

$LS_{6.1} \neq \mathcal{C}(9) + \mathcal{C}(11) + \mathcal{C}(16)$
$LS_{6.1} \neq \mathcal{C}(9) + \mathcal{C}(12) + \mathcal{C}(15)$
$LS_{6.1} = \mathcal{C}(9) + \mathcal{C}(9) + \mathcal{C}(9) + \mathcal{C}(9)$ (by Theorem 2.1)

$LS_{6.1} =$

					1
				1	2
			1	2	3
5	6				
6					

$\mathcal{C}(9)$

$+$

			4		6
2		4		6	
3					
	5	6			
		1	2		
	1		3		5

$\mathcal{C}(13)$

$+$

1	2	3		5	
	3		5		
	4	5	6		
4					
				3	4
		2		4	

$\mathcal{C}(14)$

$LS_{6.1} =$

					1
		5	6		
4	5	6			
5				3	4
			3	4	

$\mathcal{C}(11)$

$+$

1		3	4		
		4	5	6	
3					2
				2	
		1			
	1				

$\mathcal{C}(11)$

$+$

	2			5	6
2	3				
	4			1	
			1		3
	6		2		
6		2			5

$\mathcal{C}(14)$

$LS_{6.1}$ =

					1
				1	2
		6	1	2	
5	6				
			3	4	5

$\mathcal{C}(11)$

+

		3		5	6
		4	5		
		5	6		
4					
			2	3	
6	1				

$\mathcal{C}(12)$

+

1	2		4		
2	3			6	
3	4				
	5				3
		1			4
		2			

$\mathcal{C}(13)$

$LS_{6.1}$ =

					1
				1	2
			1	2	3
	6	1	2	3	
6					5

$\mathcal{C}(12)$

+

				5	6
2			5	6	
3	4	5			
4		6			
					4
	1				

$\mathcal{C}(12)$

+

1	2	3	4		
	3	4			
			6		
	5				
5					
		2	3	4	

$\mathcal{C}(12)$

Latin square 6.2 = $LS_{6.2}$

$LS_{6.2}$ =

					6
	1		3		
		5	6		
4				2	
				4	3
	5		1		

$\mathcal{C}(11)$

+

1	2	3			
		4			5
3				1	
			5		
	6				
		2			4

$\mathcal{C}(11)$

+

			4	5	
2				6	
	4				2
	3	6			1
5		1	2		
6				3	

$\mathcal{C}(14)$

$LS_{6.2}$ =

	1		3		5
			6	1	
	3	6		2	
5					
		2			4

$\mathcal{C}(11)$

+

1	2	3	4		
				6	
	4				2
4			5		
	6				3
			1		

$\mathcal{C}(12)$

+

				5	6
2		4			
3		5			
					1
		1	2	4	
6	5			3	

$\mathcal{C}(13)$

$LS_{6.2}$ =

	1		3		5
				1	2
	3	6			
5			2		
6		2			4

$\mathcal{C}(12)$

+

	2		4		6
		4			
3	4				
			5	2	
		1			3
	5			3	

$\mathcal{C}(12)$

+

1		3		5	
2				6	
		5	6		
4					1
	6			4	
			1		

$\mathcal{C}(12)$

Latin square 6.3 = $LS_{6.3}$

$LS_{6.3} \neq \mathcal{C}(11) + \mathcal{C}(11) + \mathcal{C}(14)$

$LS_{6.3}$ =

				5	6
	1		3		
		1			
4					2
	4			1	
6		2			

$\mathcal{C}(11)$

+

1	2				
		4		6	
3				2	4
		5	1		
					3
	3		5		

$\mathcal{C}(12)$

+

		3	4		
2					5
	5		6		
	6			3	
5		6	2		
				4	1

$\mathcal{C}(13)$

$LS_{6.3} = \mathcal{C}(12) + \mathcal{C}(12) + \mathcal{C}(12)$

					6
	1		3		
		1		2	
		5	1		
5			2		
6		2		4	

$\mathcal{C}(12)$

$+$

	2		4		
				6	5
3	5				
4				3	
		6		1	
	3				1

$\mathcal{C}(12)$

$+$

1		3		5	
2		4			
			6		4
	6				2
	4				3
			5		

$\mathcal{C}(12)$

Latin square 6.4 = $LS_{6.4}$

$LS_{6.4} \neq \mathcal{C}(10) + \mathcal{C}(10) + \mathcal{C}(16)$
$LS_{6.4} \neq \mathcal{C}(10) + \mathcal{C}(11) + \mathcal{C}(15)$

$LS_{6.4} = \mathcal{C}(10) + \mathcal{C}(12) + \mathcal{C}(14)$

1					6
			3		
4	6				
			2	3	
6		2			4

$\mathcal{C}(10)$

$+$

	2	3		5	
2					
	5		6	4	
		5	1		3
	4				
			5		

$\mathcal{C}(12)$

$+$

			4		
	1	4		6	5
3		1			2
				2	
5		6			1
	3			1	

$\mathcal{C}(14)$

$LS_{6.4} = \mathcal{C}(10) + \mathcal{C}(13) + \mathcal{C}(13)$

			4	5	
2					
3					2
		5	1		
	3		5	1	

$\mathcal{C}(10)$

$+$

		3			
	1			6	5
		1	6	4	
4				2	
5	4				1
6					

$\mathcal{C}(13)$

$+$

1	2				6
		4	3		
	5				
	6				3
		6	2	3	
		2			4

$\mathcal{C}(13)$

$LS_{6.4} = \mathcal{C}(11) + \mathcal{C}(11) + \mathcal{C}(14)$

					6
	1		3		
3	5				
4					
			2	3	
		2	5		4

$\mathcal{C}(11)$

$+$

		3	4		
2				6	
			6		2
			1	2	
	4				
6				1	

$\mathcal{C}(11)$

$+$

1	2			5	
		4			5
		1		4	
	6	5			3
5		6			1
	3				

$\mathcal{C}(14)$

$LS_{6.4} = \mathcal{C}(11) + \mathcal{C}(12) + \mathcal{C}(13)$

	1		3		5
		1			2
			1		3
	4	6			
6		2			

$\mathcal{C}(11)$

$+$

		3		5	6
2		4			
			6	4	
4		5		2	
			2		
	3				

$\mathcal{C}(12)$

$+$

1	2		4		
				6	
3	5				
	6				
5				3	1
			5	1	4

$\mathcal{C}(13)$

$LS_{6.4} = \mathcal{C}(12) + \mathcal{C}(12) + \mathcal{C}(12)$

	1		3		5
		1			2
			1		3
	4	6			
6	3		5		

$\mathcal{C}(12)$

$+$

		3			6
2				6	
			6	4	
4		5			
			2	3	
				1	4

$\mathcal{C}(12)$

$+$

1	2		4	5	
		4			
3	5				
	6			2	
5					1
		2			

$\mathcal{C}(12)$

Latin square 6.5 = $LS_{6.5}$

$LS_{6.5} \neq \mathcal{C}(10) + \mathcal{C}(10) + \mathcal{C}(16)$
$LS_{6.5} \neq \mathcal{C}(10) + \mathcal{C}(11) + \mathcal{C}(15)$

$LS_{6.5}$ = $\mathcal{C}(10)$ + $\mathcal{C}(12)$ + $\mathcal{C}(14)$

		3			6
2	1				5
		6		3	
5			1		
			5		

$\mathcal{C}(10)$

	2		4		
				6	
3			6		
4					1
		2			3
6	3	1			

$\mathcal{C}(12)$

1				5	
		4	3		
	4	5		1	2
	5		2		
	6			4	
				2	4

$\mathcal{C}(14)$

$LS_{6.5}$ = $\mathcal{C}(10)$ + $\mathcal{C}(13)$ + $\mathcal{C}(13)$

		3	4		
	1			6	5
			2	3	
5	6				
6					

$\mathcal{C}(10)$

					6
2			3		
3		5	6		
					1
		2		4	
	3	1	5		4

$\mathcal{C}(13)$

1	2			5	
		4			
	4			1	2
4	5	6			
			1		3
				2	

$\mathcal{C}(13)$

$LS_{6.5}$ = $\mathcal{C}(11)$ + $\mathcal{C}(11)$ + $\mathcal{C}(14)$

	1		3		5
3			6		
		6			
		2		4	
6	3		5		

$\mathcal{C}(11)$

		3	4		
2				6	
	4				
	5			3	
5	6				
		1			4

$\mathcal{C}(11)$

1	2			5	6
		4			
		5		1	2
4			2		1
			1		3
				2	

$\mathcal{C}(14)$

$LS_{6.5}$ = $\mathcal{C}(11)$ + $\mathcal{C}(12)$ + $\mathcal{C}13$

	1		3		5
				1	2
4		6			
			1		3
6				2	
		3		5	

$\mathcal{C}(11)$

2		4			
3			6		
	5			3	
	6			4	
		1			4
1	2		4		6

$\mathcal{C}(12)$

				6	
	4	5			
			2		1
5		2			
	3		5		

$\mathcal{C}13$

$LS_{6.5}$ = $\mathcal{C}(12)$ + $\mathcal{C}(12)$ + $\mathcal{C}(12)$

	1		3		5
				1	2
		6		3	
5					3
6			5		4

$\mathcal{C}(12)$

		3		5	6
2					
	4		6		
4	5				
	6	2			
		1		2	

$\mathcal{C}(12)$

1	2		4		
		4		6	
3		5			
			2		1
			1	4	
	3				

$\mathcal{C}(12)$

Latin square 6.6 = $LS_{6.6}$

$LS_{6.6}$ = $\mathcal{C}(11)$ + $\mathcal{C}(11)$ + $\mathcal{C}(14)$

				5	6
	1		3		
	4				
4					
	6	1			4
		2	5		

$\mathcal{C}(11)$

1	2				
		4		6	
		5	6		
	5				3
	3			4	1

$\mathcal{C}(11)$

		3	4		
2					5
3				1	2
		6	1	2	
5			2	3	
6					

$\mathcal{C}(14)$

$LS_{6.6} = $

					6
	1		3		
				1	
4	5				
5	6				4
		2		4	

$\mathcal{C}(11)$

$+$

	2		4	5	
		4			
3		5			2
			1		3
				3	
6					1

$\mathcal{C}(12)$

$+$

1		3			
2				6	5
	4		6		
		6		2	
		1	2		
	3		5		

$\mathcal{C}(13)$

$LS_{6.6} = $

					6
	1		3		
				1	
			1	2	3
5			2		
6		2		4	

$\mathcal{C}(12)$

$+$

	2	3	4		
				6	
3	4	5	6		
4	5				
					4
					1

$\mathcal{C}(12)$

$+$

1				5	
2		4			5
					2
		6			
	6	1		3	
	3		5		

$\mathcal{C}(12)$

Latin square 6.7 $= LS_{6.7}$

$LS_{6.7} \neq \mathcal{C}(18) + \mathcal{C}(18)$

$LS_{6.7} = $

					6
		1	6		5
	1	2			
4				2	
5			3		
	4			3	

$\mathcal{C}(12)$

$+$

	2		4	5	
2	3				
			5		
		6			3
		4		1	
6					1

$\mathcal{C}(12)$

$+$

1		3			
				4	
3				6	4
	5		1		
	6				2
		5	2		

$\mathcal{C}(12)$

Latin square 6.8 $= LS_{6.8}$

$LS_{6.8} \neq \mathcal{C}(11) + \mathcal{C}(11) + \mathcal{C}(14)$

$LS_{6.8} = $

				5	
	1		3		
3					4
	6	2			
5				4	
			1		2

$\mathcal{C}(11)$

$+$

1	2	3	4		
				6	5
			6	2	
4					
	3		2		
		5			

$\mathcal{C}(12)$

$+$

					6
2		4			
	5	1			
			5	1	3
		6			1
6	4			3	

$\mathcal{C}(13)$

$LS_{6.8} = $

	1		3		5
		1		2	4
4	6				
	3	6			
				3	2

$\mathcal{C}(12)$

$+$

		3	4	5	
2				6	
	5				
		2			3
5					1
	4		1		

$\mathcal{C}(12)$

$+$

1	2				6
		4			
3			6		
			5	1	
			2	4	
6		5			

$\mathcal{C}(12)$

Latin square 6.9 $= LS_{6.9}$

$LS_{6.9} \neq \mathcal{C}(10) + \mathcal{C}(10) + \mathcal{C}(16)$
$LS_{6.9} \neq \mathcal{C}(10) + \mathcal{C}(11) + \mathcal{C}(15)$

$LS_{6.9}$ =

	2				
			3		5
3					4
	6	2			
5				1	3

$\mathcal{C}(10)$

+

1					6
	1	4			
	5		6		
				3	
		6	2		
	3	5		4	

$\mathcal{C}(12)$

+

		3	4	5	
2				6	
		1		2	
4			5		1
	4				
6			1		2

$\mathcal{C}(14)$

$LS_{6.9}$ =

					6
	1	4	3		
	5	1			
	4			1	
6					2

$\mathcal{C}(10)$

+

			4	5	
2					
3				2	4
4			5		1
5		6	2		
	3				

$\mathcal{C}(13)$

+

1	2	3			
				6	5
			6		
	6	2		3	
					3
		5	1	4	

$\mathcal{C}(13)$

$LS_{6.9}$ =

					6
	1	4			
	5	1		2	
	4			1	3
6					2

$\mathcal{C}(11)$

+

	2	3			
				6	
			6		4
4					1
5					
	3	5	1		

$\mathcal{C}(11)$

+

1			4	5	
2			3		5
3					
	6	2	5	3	
		6	2		
				4	

$\mathcal{C}(14)$

$LS_{6.9}$ =

					6
	1		3		
		1		2	
4			5		
	4		2	1	
6					

$\mathcal{C}(11)$

+

1	2	3			
				6	
	5				4
		2			1
5		6			
	3			4	

$\mathcal{C}(12)$

+

			4	5	
2		4			5
3			6		
	6			3	
					3
		5	1		2

$\mathcal{C}(13)$

$LS_{6.9}$ =

	1		3		5
		1		2	4
	6	2			
	4			1	3
6					

$\mathcal{C}(12)$

+

			4	5	6
2					
3	5				
			5		1
		6			
	3	5		4	

$\mathcal{C}(12)$

+

1	2	3			
		4		6	
			6		
4				3	
5			2		
			1		2

$\mathcal{C}(12)$

Latin square 6.10 = $LS_{6.10}$

$LS_{6.10} \neq \mathcal{C}(10) + \mathcal{C}(10) + \mathcal{C}(16)$
$LS_{6.10} \neq \mathcal{C}(10) + \mathcal{C}(11) + \mathcal{C}(15)$
$LS_{6.10} \neq \mathcal{C}(10) + \mathcal{C}(12) + \mathcal{C}(14)$
$LS_{6.10} \neq \mathcal{C}(10) + \mathcal{C}(13) + \mathcal{C}(13)$

$LS_{6.10}$ =

					6
2			3		
		1			
4					
	3			1	4
		2	5	3	

$\mathcal{C}(11)$

+

		3	4		
	1			6	
3				4	
	6		1	2	
5					
					1

$\mathcal{C}(11)$

+

1	2			5	
		4			5
	5		6		2
		5			3
		6	2		
6	4				

$\mathcal{C}(14)$

$LS_{6.10}$ =

					6
	1		3		
		1			
4					
5		6	2		
	4		5	3	

$\mathcal{C}(11)$

+

1		3	4		
2					5
			6	4	2
		5			
	3			1	
6					

$\mathcal{C}(12)$

+

	2			5	
		4		6	
3	5				
	6		1	2	3
					4
		2			1

$\mathcal{C}(13)$

$LS_{6.10}$ =

	1		3		5
		1			2
			1		3
5		6			
6	4			3	

$\mathcal{C}(12)$

+

		3	4		6
2					
	5		6		
4				2	
				1	4
		2	5		

$\mathcal{C}(12)$

+

1	2			5	
		4		6	
3				4	
	6	5			
	3		2		
					1

$\mathcal{C}(12)$

Latin square 6.11 = $LS_{6.11}$

$LS_{6.11} \neq \mathcal{C}(10) + \mathcal{C}(10) + \mathcal{C}(16)$
$LS_{6.11} \neq \mathcal{C}(10) + \mathcal{C}(11) + \mathcal{C}(15)$

$LS_{6.11}$ =

				5	
2					
				1	5
4	6		2		
					4
6			3		

$\mathcal{C}(10)$

+

1	2	3	4		
			5		
3	4				
		5			1
	3			2	
				4	

$\mathcal{C}(12)$

+

					6
	1	4		6	3
		2	6		
				3	
5		6	1		
	5	1			2

$\mathcal{C}(14)$

$LS_{6.11}$ =

					6
2	1				
	4			1	
				3	
	3		1		
6					2

$\mathcal{C}(10)$

+

	2				
				6	3
3		2			
4		5	2		
5		6			
	5	1		4	

$\mathcal{C}(13)$

+

1		3	4	5	
		4	5		
			6		5
	6				1
				2	4
			3		

$\mathcal{C}(13)$

$LS_{6.11}$ =

	1				3
				1	5
			2	3	1
5		6			
6					2

$\mathcal{C}(11)$

+

1			4		
		4		6	
3					
4		5			
	3			2	
		1	3		

$\mathcal{C}(11)$

+

	2	3		5	6
2			5		
	4	2	6		
	6				
			1		4
	5			4	

$\mathcal{C}(14)$

$LS_{6.11}$ =

	1				3
				1	5
			2	3	1
5		6			
6					2

$\mathcal{C}(11)$

+

1		3	4		6
			5	6	
3		2			
			1		4
	5			4	

$\mathcal{C}(12)$

+

	2			5	
2		4			
	4		6		
4	6	5			
	3			2	
		1	3		

$\mathcal{C}(13)$

$LS_{6.11} =$

	1				3
				1	5
	6		2	3	
			1	2	4
6	5				

$\mathcal{C}(12)$

$+$

			4	5	6
2				6	
		2			
4		5			
	3	6			
			3	4	

$\mathcal{C}(12)$

$+$

1	2	3			
		4	5		
3	4		6		
					1
5					
		1			2

$\mathcal{C}(12)$

Latin square 6.12 $= LS_{6.12}$

$LS_{6.12} =$

		1			5
	1	2		6	
				3	
		4		2	3
6			3		

$\mathcal{C}(11)$

$+$

	2	3			
			6		
3					4
	5	6			
	6		1		
		5			2

$\mathcal{C}(11)$

$+$

1			4	5	6
2	3			4	
			5		
4			2		1
5					
	4			1	

$\mathcal{C}(14)$

$LS_{6.12} =$

		1			5
	1	2		6	
				3	
		4		2	3
6			3		

$\mathcal{C}(11)$

$+$

1		3			6
			6		
3			5		
	5				
5	6		1		
	4				2

$\mathcal{C}(12)$

$+$

	2		4	5	
2	3			4	
					4
4		6	2		1
		5		1	

$\mathcal{C}(13)$

$LS_{6.12} =$

		1			5
	1	2		6	
				3	
		4		2	3
6	4	5			

$\mathcal{C}(12)$

$+$

		3		5	6
2	3		6		
					4
4					1
	6		1		
			3		

$\mathcal{C}(12)$

$+$

1	2		4		
				4	
3			5		
	5	6	2		
5					
				1	2

$\mathcal{C}(12)$

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