Defining sets which intersect each Latin trade at least twice

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Abstract

A defining set of a Latin square is a partially filled-in Latin square which completes to no other Latin square of the same order. We introduce the concept of a k-strong defining set, in which if any k entries are deleted, the property of being a defining set is retained. Equivalently, a k-strong defining set intersects every Latin trade in the Latin square at least k times. In the addition table for integers modulo n, when n is even we determine the minimum size of a k-strong defining set for any k. For odd n we give a construction for a minimally 2-strong defining set. We furthermore give computational results for Latin squares of small orders.

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1 Introduction

In what follows, rows and columns of an $n \times n$ array L are each indexed by a set $N(n)$ of size n, with $L_{i,j}$ denoting the *symbol* in cell (i, j). We sometimes consider an array L to be a set of ordered triples $L = \{(i, j; L_{i,j})\}$ so that the notion of a subset of an array is precise. A *partial latin square* (PLS) of *order n* is an $n \times n$ array in which some cells may be empty, the set of possible symbols $N(n)$ has size n, and each symbol occurs at most once in each row and column. A *Latin square* is a PLS with no empty cells.

A *Latin trade* is a non-empty PLS T such that there exists another PLS T' of the same order, where: (a) $T \cap T' = \varnothing$, (b) T and T' have the same set of non-empty cells, (c) T and T' share the same set of symbols in each row; and (d) T and T' share the same set of entries in each column. We say that T' is the *disjoint mate* of the Latin trade T. In some literature the pair (T, T') are together called a *bitrade*. If we take any two distinct Latin squares L and L' of the same order, observe that $L \setminus L'$ is a Latin trade with disjoint mate $L' \setminus L$. Thus Latin trades describe the differences between a Latin square and any other square of the same order.

A *defining set* D of a Latin square L is a subset of L such that if L' is a distinct Latin square of the same order as L, D is not a subset of L'. We sometimes say that D has a *unique completion* to L. If D is minimal with respect to this property we say that D is a *critical set* of L. Necessarily, a defining set D of a Latin square L intersects every Latin trade T in L. (If not, D completes to the distinct Latin squares L and $(L \setminus T) \cup T'$.) Thus the theory of defining sets and Latin trades are very much intertwined.

We define B_n to be the operation table for the integers modulo n. That is,

$$B_n := \{(i, j; i + j \pmod n) : i, j \in \mathbb{Z}_n\}.$$

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This particular Latin square may have some extremal properties in terms of Latin trades and critical sets, which we now outline. Let p be the smallest prime dividing n. The size of the smallest latin trade in B_n was shown to be at least $e \log p + 3$ [10]. Conversely, in [12] it is shown that for all n, B_n contains a Latin trade of size at most $5 \log_2 p$. The smallest possible size of a Latin trade is 4, given by any 2×2 subsquare. Such Latin trades are called *intercalates*. Meanwhile, any Latin square has a Latin trade of size at most $8\sqrt{n}$ [5]. It is conjectured in [5] that if L is a Latin square of order prime p, the smallest Latin trade in L is no larger than the smallest Latin trade in B_p.

Let $\mathrm{scs}(L)$ be the size of the smallest critical set in a Latin square L and let $\mathrm{scs}(n)$ in turn be the size of the smallest critical set amongst all Latin squares of order n. In [11] it was shown that $\mathrm{scs}(n) \geqslant n^2/10^4$ for sufficiently large n. Conversely, it has been shown that $\mathrm{scs}(n) \leqslant \lfloor n^2/4 \rfloor$ for all $n \geqslant 1$ [8, 7]. Moreover, it is conjectured that equality holds in this bound, a conjecture that we know is true when $n \leqslant 8$ [3]. Interestingly, examples of critical sets of size $\lfloor n^2/4 \rfloor$ are, in the present state of the literature, known only to exist in B_n [8, 7].

We next introduce an idea which measures the amount of information in a defining set that can be "lost" while retaining the property of ebing a defining set. We say that a defining set D of a Latin square L is *k-strong* if for every trade $T \subseteq L$, $|D \cap T| \geqslant k$. In turn, a defining set D is said to be *minimally k-strong* if it is k-strong but any strict subset of D is not. By definition, any defining set is 1-strong and any critical set is minimally 1-strong.

Since the number of trades in a Latin square of order n is one less than the number of Latin squares of order n, verifying that a subset is k-strong by checking every trade can be cumbersome. The following lemma gives a more efficient method, which we make use of in Section 3.

Lemma 1. *Let $D \subset L$ where L is a Latin square. Then D is a k-strong defining set if and only if for any $D' \subset D$ such that $|D'| < k$, $D \setminus D'$ is a defining set. A k-strong defining set D is, in turn, minimally k-strong if and only if for each triple $(i, j; L_{i,j}) \in D$ there exists a Latin trade $T \subseteq L$ such that $(i, j; L_{i,j}) \in T$ and $|T \cap D| = k$.*

Proof. Suppose first that D is minimally k-strong. Then by definition, D intersects every Latin trade in L at least k times. In turn, for any $D' \subseteq D$ such that $|D'| < k$, $D \setminus D$ intersects every Latin trade in L at least once, and is thus a defining set. Conversely, suppose that D is a subset of a Latin square such that for any $D' \subset D$ such that $|D'| < k$, $D \setminus D'$ is a defining set. Suppose there exists a trade $T \subseteq L$ such that $|T \cap D| < k$. Let T' be a disjoint mate of T. Then letting $D' = T \cap D$, $D \setminus D'$ is a subset of the Latin square $(L \setminus T) \cap T' \neq L$. Therefore $D \setminus D'$ is not a defining set for L, a contradiction. Thus (a) is satisfied.

Next, suppose that D is k-strong but not minimally so. Then there exists $(i, j; L_{i,j}) \in D$ such that $D \setminus \{(i, j; L_{i,j})\}$ is k-strong. By definition, any trade T intersects $D \setminus \{(i, j; L_{i,j})\}$ at least k times. Therefore for any trade T such that $\{(i, j; L_{i,j})\} \in T$, $|T \cap D| > k$. Next, suppose that D is minimally k-strong. Then for any $(i, j; L_{i,j}) \in D$, $D' = D \setminus \{(i, j; L_{i,j})\}$ is not k-strong. Thus there exists a Latin trade $T \subseteq L$ such that $|T \cap D| \geqslant k$ but $|T \cap D'| < k$. Thus $\{(i, j; L_{i,j})\} \in T$. □

Observe the following.

Lemma 2. *Let d be the size of the smallest Latin trade in a given Latin square L. Then L itself is d-strong.*

Note that L is only in turn minimally d-strong if every element of L belongs to a Latin trade of size d. For example, there is a Latin square of order 5 which contains an intercalate but for which not every element is in an intercalate. The following implies that if d is the size of the smallest latin trade in a Latin square L, we can find a chain of PLSs $D_1 \subset D_2 \subset \cdots \subset D_d \subseteq L$ such that D_k is minimally k-strong for $1 \leqslant k \leqslant d$.

Lemma 3. *Let P be a minimally k-strong subset of a Latin square L of order $n \geqslant 2$, where $k \geqslant 1$. Then there is a* PLS *$Q \subset P$ such that Q is minimally $(k-1)$-strong.*

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

Figure 1: The PLS P_{11}

Proof. If $k = 1$, then $Q = \varnothing$, since every element of a Latin square belongs to some trade (such as that formed by swapping two rows).

Otherwise, for any $(i, j; k) \in P$, by definition, $Q' = P \setminus \{(i, j; k)\}$ is not k-strong. On the other hand, every trade in L intersects Q' at least $k - 1$ times. For each element of Q', recursively remove that element if and only if the remaining PLS is $(k-1)$-strong. The final result of this algorithm is a minimally $(k-1)$-strong PLS $Q \subseteq Q' \subset P$. $\square$

For any $S \subseteq B_n$ and integers a and b, we define

$$S \oplus (a, b) := \{(i + a \pmod n), j + b \pmod n); k + a + b \pmod n)) : (i, j; k) \in S)\}.$$

Observe that $S \oplus (a, b) \subseteq B_n$. Let T_n be a Latin trade in B_n of minimum size d_n. Since $T_n \oplus (a, b)$ is also a Latin trade in B_n for any integers a and b, it immediately follows that The Latin square B_n is itself a minimal d_n-strong defining set.

The main results of this paper are as follows. Let $\mathrm{sds}(L, k)$ be the size of the smallest k-strong defining set in the Latin square L. In Section 2, we prove the following.

Theorem 4. *If n is even,* $\mathrm{sds}(B_n, k) = kn^2/4$ *where* $1 \leqslant k \leqslant 4$.

Next, for each $i \in \mathbb{Z}_n$, we define the *diagonal* $D_i \subset B_n$ to be the PLS

$$\{(r, r + i; 2r + i \pmod n)) : r \in \mathbb{Z}_n\}.$$

In Section 3, we prove the following.

Theorem 5. *The* PLS *given by* $P_n := D_0 \cup D_1 \cup \cdots \cup D_{\lceil (n-3)/2 \rceil}$ *is a minimally* 2*-strong defining set of* B_n *for all* $n \geqslant 2$.

See Figure 1 for an example of P_{11}. In Section 3 we give computational results for small orders. Finally in Section 4, we describe k-strong defining sets in B_n for larger values of k.

2 Disjoint critical sets in Latin squares and the even case

In [1], the question of whether a Latin square can be partitioned into disjoint critical sets is studied. This is of interest to our problem, because if $C_1, \ldots, C_m$ are disjoint critical sets in a Latin square L, then, since each trade must intersect each of these critical sets, the union $C_1 \cup C_2 \cup \cdots \cup C_m$ is an m-strong defining set in L. In [1], it is shown, in particular, that B_n can be partitioned into 4 disjoint critical sets for any integer n. For n even, this result allows us to determine $\mathrm{sds}(B_n, k)$ exactly, as we now show.

Theorem 6. [1] *For any even $n \geqslant 2$, B_n partitions into the following 4 critical sets, each of size $n^2/4$:*

$$C_1 := \{(i,j;i+j) : 0 \leqslant i,j;\ \ i+j \leqslant n/2-1\} \cup \{(i,j;i+j) : n/2+1 \leqslant i+j;\ \ i,j \leqslant n-1\};$$

$$C_2 := C_1 \oplus (0,n/2); C_3 := C_1 \oplus (n/2,0); C_4 := C_1 \oplus (n/2,n/2).$$

Since, by definition, any Latin trade intersects each critical set at least once, it follows that for $1 \leqslant k \leqslant 4$, any Latin trade intersects $C_1 \cup \cdots \cup C_k$ at least k times. Now, define $I_{i,j}$ to be the intercalate in B_n on cells (i,j), $(i+n/2,j)$, $(i,j+n/2)$ and $(i+n/2,j+n/2)$. Each intercalate of this form intersects each of C_1, C_2, C_3 and C_4 exactly once. Moreover, the set of intercalates $\mathcal{I} := \{I_{i,j} \mid 0 \leqslant i,j \leqslant n/2-1\}$ partition B_n. This implies Theorem 4.

3 Tessellations of triangles and Latin trades in B_n

In what follows, a triangle in the Euclidean plane is said to be *good* if it is right-angled with integer coordinates and a hypotenuse of gradient -1 (thus isoceles). A tessellation of any finite region in the Euclidean plane into good triangles is also said to be *good* if no point is the vertex of more than 3 of these triangles. We also define $\mathcal{E}_n$ to be the good triangle in the Euclidean plane with vertices $(0,0)$, $(0,n)$ and $(n,0)$.

Theorem 7. [9] *Let S be a good tessellation of $\mathcal{E}_n$ (with more than one triangle) and let V be the set of coordinates of all vertices of the triangles in S. Then*

$$T := \{(i,j) : (i,j) \in S\} \setminus \{(0,n),(n,0)\}$$

is a Latin trade in B_n.

Proof. This result is proved in [9], but we add a short explanation here to be helpful. Consider a triangle Δ in the tessellation with vertex (i,j) at the right angle. Then $(i,j+k)$ and $(i+k,j)$ are the other vertices of Δ, where k is some non-zero integer. We place $i+j+k \pmod n$ in the cell (i,j) of the disjoint mate of T'. Since any point is the vertex of at most 3 triangles in the tessellation, it follows that (i,j) is not the point at the right-angle of any other triangle in the tessellation. Thus T' is well-defined. Moreover, since $k \neq 0$, T and T' are disjoint. By similar reasoning, it can also be shown that T' and T contain the same set of symbols in each row and column. □

An example of the previous theorem can be seen in Figure 3. Note that (x,y)-coordinates in the plane become visually transposed in the Latin square. That is, the row and column are, visually, the $-y$ and x axes, if we consider the origin to be the top-left corner of the Latin square. This is a classic example of the confusion when switching between the Euclidean plane and an array. The triangle in Figure 3 has thus been reflected on the x-axis so that the trade in B_{11} can be better visualized. The triangles in the tessellation of $\mathcal{E}_{11}$ are given by:

$$\{(0,0),(0,4),(4,0)\},\{(4,4),(0,4),(4,0)\},\{(0,4),(1,4),(0,5)\},\{(0,4),(1,4),(1,5)\},$$
$$\{(0,5),(1,5),(0,6)\},\{(0,5),(1,5),(1,6)\},\{(0,6),(1,6),(0,7)\},\{(0,6),(1,6),(1,7)\},$$
$$\{(1,4),(4,4),(1,7)\},\{(4,7),(4,4),(1,7)\},\{(0,7),(0,11),(4,7)\},\{(4,0),(4,7),(11,0)\}.$$

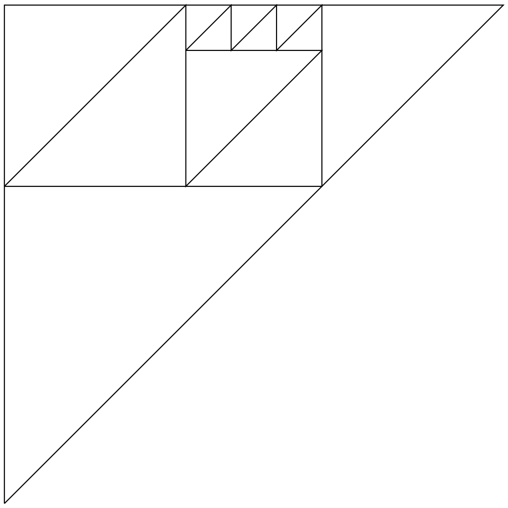
The following lemma follows by recursively tessellating into maximal squares, which each tessellate into pairs of isoceles right-angled triangles.

Lemma 8. *There is a good tessellation of any rectangle in the Euclidean plane with integer coordinates.*

Lemma 9. *Let $2 \leqslant 2m < n < 3m$. Then there is a Latin trade $T_{m,n}$ in B_n such that:*

$$T_0 := \{(0,0;0),(m,0;m),(m,m;2m),(m,n-m;0)\} \subseteq T$$

and if $(r,c;r+c) \in T_{m,n} \setminus T_0$, then $0 \leqslant r \leqslant 3m-n$ and $m \leqslant c \leqslant n-m$.



$\mathbf{0_4}$	1	2	3	$\mathbf{4_5}$	$\mathbf{5_6}$	$\mathbf{6_7}$	$\mathbf{7_0}$	8	9	10
1	2	3	4	$\mathbf{5_8}$	$\mathbf{6_5}$	$\mathbf{7_6}$	$\mathbf{8_7}$	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
$\mathbf{4_0}$	5	6	7	$\mathbf{8_4}$	9	10	$\mathbf{0_8}$	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

Figure 2: A good tessellation of $\mathcal{E}_{11}$ (reflected on the x-axis) and the correpsonding Latin trade in B_{11}

Proof. Consider the tessellation of $\mathcal{E}_n$ given by the triangles and rectangle with the following vertex sets: $\{(0,0),(0,m),(m,0)\}$, $\{(m,m),(0,m),(m,0)\}$, $\{(m,0),(m,n-m),(n,0)\}$, $\{(0,n-m),(m,n-m),(0,n)\}$, $\{(3m-n,m),(3m-n,n-m),(m,m)\}$, $\{(m,n-m),(3m-n,n-m),(m,m)\}$, $\{(0,m),(0,n-m),(3m-n,m),(3m-n,n-m)\}$. The result then follows by Theorem 7 and Lemma 8. □

The trade in Figure 3 is in fact an example of the previous lemma with $n = 11$ and $m = 3$.

Lemma 10. *Let $n > 3m \geqslant 3$. Then there is a Latin trade $T_{m,n}$ in B_n such that:*

$$T_0 := \{(0,0;0),(m,0;m),(m,m;2m),(0,m;m)\} \subseteq T$$

and if $(r,c;r+c) \in T_{m,n} \setminus T_0$, then $0 \leqslant r \leqslant m$ and $2m \leqslant c \leqslant n-m$.

Proof. Consider the tessellation of $\mathcal{E}_n$ given by the triangles and rectangle with the following vertex sets: $\{(0,0),(0,m),(m,0)\}$, $\{(m,m),(0,m),(m,0)\}$, $\{(0,m),(0,2m),(m,m)\}$, $\{(m,2m),(0,2m),(m,m)\}$, $\{(0,n-$

0	1	2	$\mathbf{3_7}$	4	5	6	$\mathbf{7_8}$	$\mathbf{8_9}$	$\mathbf{9_{10}}$	$\mathbf{10_3}$
1	2	3	4	5	6	7	$\mathbf{8_0}$	$\mathbf{9_8}$	$\mathbf{10_9}$	$\mathbf{0_{10}}$
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	$\mathbf{7_3}$	8	9	10	$\mathbf{0_7}$	1	2	$\mathbf{3_0}$
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

Figure 3: A Latin trade in B_{11} that intersects P_{11} twice.

0	1	$\mathbf{2_5}$	3	4	$\mathbf{5_8}$	6	7	$\mathbf{8_9}$	$\mathbf{9_{10}}$	$\mathbf{10_2}$
1	2	3	4	5	6	7	8	$\mathbf{9_0}$	$\mathbf{10_9}$	$\mathbf{0_{10}}$
2	3	4	5	6	7	8	9	10	0	1
3	4	$\mathbf{5_2}$	6	7	$\mathbf{8_5}$	9	10	$\mathbf{0_8}$	1	$\mathbf{2_0}$
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

Figure 4: Another Latin trade in B_{11} that intersects P_{11} twice.

$m), (0,n), (m,n-m)\}$, $\{(m,0),(m,n-m),(n,0)\}$, $\{(0,2m),(0,n-m),(m,2m),(m,n-m)\}$. The result then follows by Theorem 7 and Lemma 8. □

We will proceed to prove Theorem 5 via the use of Lemma 1.

Lemma 11. *Let $n \geqslant 2$. For each element $(i,j;k)$ of P_n, there exists a Latin trade $T \subset B_n$ such that $(i,j;k) \in T$ and $|T \cap B_n| = 2$.*

Proof. Since $P_n \oplus (1,1) = P_n$, without loss of generality, we may assume that $i = 0$. We first consider when n is even. Then observe that the Latin trade

$$T := \{(0,j;j),(0,j+n/2,j+n/2),(n/2,j;j+n/2),(n/2,j+n/2;j)\}$$

intersects the diagonal D_j and $D_{j+n/2}$ twice each. Since $n/2-1 < j+n/2 \leqslant n-1$, the diagonal $D_{j+n/2}$ does not intersect P_n. Thus T intersects P_n exactly twice.

Otherwise n is odd. Observe that $P_n^T \oplus ((n+3)/2, 0) = P_n$. Thus we can assume that $\lceil (n-3)/4 \rceil \leqslant j \leqslant (n-3)/2$.

Case 1: $\lceil (n-3)/4 \rceil \leqslant j < (n-3)/3$. By Lemma 10, there is a Latin trade $T_{j+1,n} \oplus (0,j)$ in B_n which includes $(0,j;j),(0,2j+1;2j+1),(j+1,j;2j+1),(j+1,2j+1,3j+2)$ and all other elements in cells (r,c) where $0 \leqslant r \leqslant j+1$ and $3j+2 \leqslant c \leqslant n-1$. Observe that $T_{j+1,n} \oplus (0,j)$ intersects D_j twice (at $(0,j;j)$ and $(j+1,j;2j+1)$), with all other elements in D_α for some α such that

$$(n-3)/2 < 2j+1 \leqslant \alpha \leqslant n-1.$$

Thus this trade intersects P_n exactly twice.

Case 2: $j = (n-3)/3$. Then there is a Latin trade

$$T = \{(0,j;j),(0,2j+1;2j+1),(j+1,j;2j+1),(j+1,2j+1;3j+2),(0,3j+2;3j+2),(j+1,3j+2;j)\}.$$

This intersects D_j, D_{2j+1} and $D_{3j+2} = D_{n-1}$ twice each. Since $2j+1 = (2n-3)/3 > (n-3)/2$, D_{2j+1} does not intersect P_n.

Case 3: $(n-3)/3 < j \leqslant (n-3)/2$. By Lemma 9, there is a Latin trade $T_{j+1,n} \oplus (0,j)$ in B_n which includes $(0,j;j),(j+1,j;2j+1),(j+1,2j+1;3j+2),(j+1,n-1;j)$ and all other elements in cells (r,c) where $0 \leqslant r \leqslant 3j+3-n$ and $2j+1 \leqslant c \leqslant n-1$. Observe that $T_{j+1,n} \oplus (0,j)$ intersects D_j twice (at $(0,j;j)$ and $(j+1,j;2j+1)$), with all other elements in D_α for some α such that

$$(n-3)/2 < n-2-j \leqslant \alpha \leqslant n-1.$$

Thus this trade intersects P_n exactly twice. □

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

Figure 5: The critical set Q_{11}

Next, as an intermediary step to proving Theorem 5, we define $Q_n \subseteq P_n$ as follows:

$$Q_n = \{(i, j; i + j \pmod n) : 1 \leqslant i \leqslant j \leqslant i + \lfloor (n-3)/2 \rfloor\}.$$

Lemma 12. *The* PLS Q_n *has unique completion to* B_n.

Proof. We give below a sequence of cells whose subsequent completion from Q_n is "forced" to obtain B_n. By "forced", we mean that we can sequence the empty cells so that when placing a symbol in cell (i, j), there is only one symbol which occurs in neither row i nor column j of the defining set so far. In what follows, let $N = \lceil (n-3)/2 \rceil$.

$$\begin{array}{l}
(1, N+1), (2, N+2), \dots, (n-(N+1), n-1), \\
(1, N+2), (2, N+3), \dots, (n-(N+2), n-1), \\
\dots, \\
(1, n-1), \\
(0, n-1), \\
(n-1, n-2), (0, n-2) \\
(n-2, n-3), (n-1, n-3), (0, n-3) \\
\dots \\
(1, 0), (2, 0), \dots, (n-1, 0), (0, 0).
\end{array}$$

□

Since $P_n = P_n \oplus (1, 1)$ and $Q_n \subset P_n$, we have the following corollary.

Corollary 13. *If* $P \subset P_n$ *such that* $|P_n \setminus P| = 1$, *then* P *is a defining set of* B_n.

The previous corollary, together with Lemmas 11 and 1, imply Theorem 5. As an extra, we next establish that Q_n is a minimally 1-strong defining set (i.e. a critical set) of B_n.

Lemma 14. *For each* $(i, j; k) \in Q_n$, *there exists a Latin trade* $T \subset B_n$ *such* $T \cap Q_n = \{(i, j; k)\}$.

Proof. First observe that

$$Q_n = \{(n-j, n-i; n-k) : (i, j, k) \in Q_n\}.$$

We can therefore assume, without loss of generality, that $i + j \leqslant n$.

So we can consider just two cases: (1) $j < n/2$; and (2) $j \geqslant n/2$ and $i + j \leqslant n$.

0	1			
1			4	
	3			
		0		2
			2	3

Figure 6: A 2-strong defining set of minimum size 9 in B_5.

For Case (1) we use the trade $T \oplus (i, i-1)$, where T is the Latin trade based on the tessellation of $\mathcal{E}_n$ with triangles on vertices $\{(0,0),(0,j-i+1),(j-i+1,0)\}$, $\{(j-i+1,j-i+1),(0,j-i+1),(j-i+1,0)\}$, $\{(0,j-i+1),(0,n),(n-j+i-1,j-i+1)\}$, $\{(n-j+i-1,0),(n,0),(n-j+i-1,j-i+1)\}$ and the rectangle on the set of vertices $\{(j-i+1,0),(j-i+1,j-i+1),(i-j-1,0)(j-i+1,n-j+i-1)\}$.

For Case (2), if n is even, use the Latin trade on cells (i,j), $(i+n/2,j)$, $(i,j+n/2)$ and $(i+n/2,j+n/2)$. Otherwise n is odd. Let $N=(n-1)/2$. We use the Latin trade on the set of cells

$$\{(i,j),(i,j-N)\} \cup \{(i+N,c),(i+N+1,c) : j-N \leqslant c \leqslant j\}.$$

□

Corollary 15. *The* PLS Q_n *is a critical set of* B_n *for all* $n \geqslant 2$.

4 Computational results

Although we have proven that P_n is a minimally 2-strong defining set for B_n, in general for odd n, it is not a minimal 2-strong defining set of minimum size. For $n=5$, we have verified by computer that the minimum such size is given by 9, an example of which can be seen in Figure 6.

k \ n	2	3	4	5	6	7	8	9	10	11
1	1	2	4	6	9	12	16	20	25	22–30
2	2	3	8	9	18	18	32	32	50	32–45
3	3	5	12	12	27	21	48	45	75	44–56
4	4	6	16	15	36	27	64	54	100	55–67
5		8		19		34		72		65–76
6		9		20		39		81		75–86
7				24		42				85–94
8				25		48				96–99
9						49				108
10										110
11										120
12										121

Table 1: Minimum k-strong defining set sizes in B_n.

Table 1 shows $\text{sds}(B_n,k)$ for $1 \leqslant k \leqslant 12$ and $1 \leqslant n \leqslant 11$. These results were obtained by using the integer programming approach described in [3].

For $n \leqslant 5$, we were able to generate all trades in the Latin square B_n. Using these trades we constructed a 0-1 integer linear program (ILP) to find the size of the minimal set in B_n that intersected every trade in at least k places. For n even, the results agree with Theorem 4. For $n > 6$ we were unable to generate all possible trades in B_n and thus some values, which are shaded, are simply lower bounds, or ranges in the case of $n = 11$.

One might conjecture from these results that if d_p is the size of the smallest Latin trade in B_p (where p is prime), then removing any entry from B_p results in a minimum $(d_p - 1)$-strong defining set. This conjecture is equivalent to asserting that any pair of entries in B_p is contained in a Latin trade of size d_p.

k \ main class	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	6.10	6.11	6.12
1	9	11	11	10	10	11	12	11	10	10	10	11
2	18	18	16	14	14	16	20	18	18	16	14	14
3	27	27	24	21	21	22	29	27	27	23	19	20
4	36	36	36	28	28	32	36	36	36	28	24	24
5												32
6												36

Table 2: Minimum k-strong defining set sizes in main classes of order 6.

Table 2 shows minimum k-strong defining set sizes for Latin squares in the main classes of order 6, using the main class numbering from [6]. Here, note that 6.12 is a main class with no intercalates (2×2 subsquares) and $\mathrm{sds}(L, 4) = 36$ implies L can be decomposed into 9 disjoint intercalates; otherwise $\mathrm{sds}(L, 4) < 36$. The results for $k = 1$ are the same as in [2] using a different numbering.

5 Minimal k-strong PLS in B_n for larger k

For any $0 \leqslant k < n$, define $Q_{n,k} = D_0 \cup D_1 \cup \cdots \cup D_k$. Note that if $k = \lfloor (n-3)/2 \rfloor$, $Q_{n,k} = Q_n$.

Theorem 16. *Let $x \leqslant n$ such that $n - 4n/x \geqslant 30$. there exists a Latin trade in B_n which intersects the* PLS *$Q_{n,n-\lceil 4n/x \rceil - 30}$ at most $10 \log_4(2x)$ times.*

Proof. Define a sequence $m_0, m_1, \ldots$ where $m_0 = (n-3)/2$ and $m_i = \lfloor (m_{i-1} - 3)/4 \rfloor$ for $i > 0$. Let $\alpha(z) = \alpha$ be the smallest integer such that $m_\alpha \leqslant n/x + 6$. Then, recursively, we have that:

$$m_\alpha \quad \leqslant \quad \frac{m_0}{4^\alpha} = \frac{(n-2)}{2 \times 4^\alpha}. \tag{1}$$

By definition, $m_{\alpha-1} > n/x + 6$. Therefore:

$$m_\alpha = \lfloor (m_{\alpha-1} - 3)/4 \rfloor \geqslant (m_{\alpha-1} - 6)/4 \quad > \quad n/4x. \tag{2}$$

Next, tessellate $\mathcal{E}_n$ into an $m_0 \times (m_0 + 3)$ rectangle and two good triangles. We will proceed to recursively tessellate (in a "good" way) this rectangle into a number of good triangles and increasingly small rectangles of dimension $2m_i \times 2(m_i + 3)$, for each $i \geqslant 0$. This process is shown precisely in Figure 5, which is directly taken from [12]. Note that at each stage we effectively tessellate an $m_i \times (m_i + 3)$ rectangle, magnifying this by a factor of 2.

At each stage, from Figure 5, at most 10 points of vertices outside the next rectangle are specified. Thus the corresponding Latin trade will have at most 10α cells outside the final $2m_\alpha \times 2(m_\alpha + 3)$ rectangle, which by Lemma 8, itself has a good tessellation.

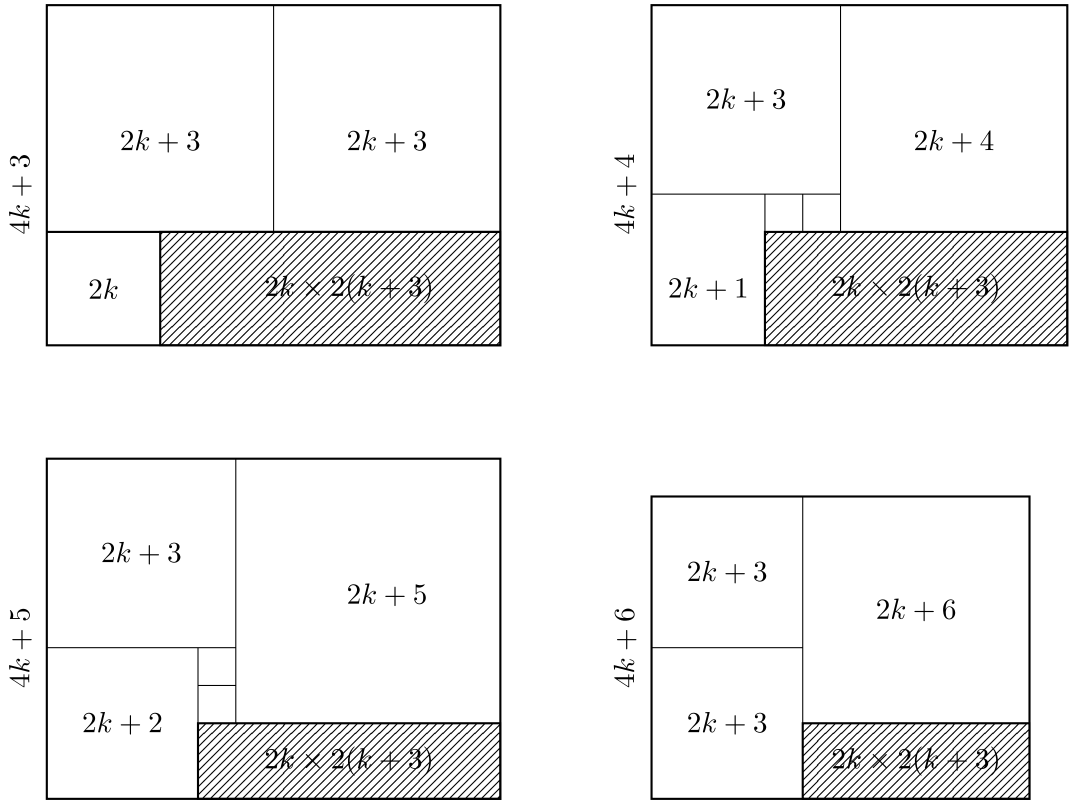


Figure 7: Figure 1 from [12]

Now, since $4m_\alpha + 6 \leqslant 4z + 6 \leqslant 4n/x + 30$, by translation of the Latin trade constructed we can position this final rectangle so that it does not intersect $Q_{n,n-\lceil 4n/x\rceil-30}$. Therefore the trade intersects $Q_{n,n-\lceil 4n/x\rceil-30}$ at most 10α times. But from (1) and (2),

$$\alpha \leqslant \log_4 n - \log_4 2m_\alpha \leqslant \log_4 n - \log_4 (n/(2x)) = \log_4(2x).$$

□

One might conjecture, given the results in this paper and the literature, that for odd n, $Q_{n,n-\lceil n/x\rceil}$ is asymptotically minimally $\Theta(\log x)$-strong. From the previous theorem, to prove such a conjecture, it would suffice to show that any Latin trade in B_n intersects $Q_{n,n-\lceil n/x\rceil}$ at least $\Theta(\log x)$ times. We leave this as an open problem.

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